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Abstract

The aim of this paper is to find approximate log-transition density functions for multivariate time-inhomogeneous diffusions in closed form. There are many empirical evidences supporting that the data generating process governing dynamics of many economics variables might vary over time because of economic climate changes or time effects. One possible way to explain the time-dependent dynamics of state variables is to model the drift or volatility terms as functions of time t as well as state variables. A way to find closed-form likelihood expansion for a multivariate time-homogeneous diffusion has been developed by Aït-Sahalia (2008). This research is built on his work and extends his results to time-inhomogeneous cases. We conduct Monte Carlo simulation studies to examine performance of the approximate transition density function when it is used to obtain ML estimates. The results reveal that our method yields a very accurate approximate likelihood function, which can be a good candidate when the true likelihood function is unavailable as is often the case.

KEY WORDS: Likelihood function; Multivariate time-inhomogeneous diffusion; Reducible diffusions, Irreducible diffusions

1 Introduction

The history of multivariate diffusion models is long (Brennan and Schwartz (1979), Langetieg (1980) and Stambaugh (1988)), whereas univariate diffusion model has been dominantly used in the literature since Merton's seminal work in the 1970s. Stochastic volatility model (Heston (1993), Andersen and Lund (1997) and Gallant and Tauchen (1998)), multifactor affine term-structure model (Duffie and Kan (1996), Balduzzi

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et al. (1996) (hereafter BDFS) and Dai and Singleton (2000)), and quadratic term-structure model (Ahn et al. (2002)) are more recent examples.

Continuous-time diffusion models written as m -dimensional stochastic differential equation (SDE) form of (1) have been widely used in many different economic areas from finance to game theory. These fields include optimal control theory (Dixit and Pindyck (1994)), contingent claim pricing and portfolio choice (Duffie (2001)), dynamics of exchange rate (Froot and Obstfeld (1991)), contract theory (Holmstrom and Milgrom (1987)) and game theory (Bolton and Harris (1999)). In these literature, researchers model many economic variables such as interest rates, asset prices, exchange rates, electricity prices, profits, revenue, costs, payoffs and so forth using diffusion processes.

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t \quad (1)$$

Equation (1) is a time-inhomogeneous diffusion model since the drift function, μ , or the volatility function, σ , depend not only on state variable vector X_t but also on time t . If they are functions of just state variables X_t it is called a time-homogeneous diffusion process.

There are many empirical evidences supporting that the data generating process governing dynamics of many economics variables might vary over time because of economic climate changes or time effects. One possible way to explain the time dependence of state variables is to model the drift or volatility terms as functions of time t as well as state variables as in equation (1). By doing so, we can let the data say whether or not there is time dependence in the data that cannot be attributable to random variability of the state variable.

Naming some of evidences for time inhomogeneity in the literature, the volatilities of interest rates at all maturities are known to increase around releases of key macroeconomic announcements and on Federal Open Market Committee (FOMC) meeting days (Bartolini et al. (2002) and Piazzesi (2005)). The former events also cause increased volatility of exchange rates (Andersen et al. (2003)). The electricity price and the return volatilities of stock price, exchange rate and US Treasury bond futures contract show intraday periodicity (Misiorek et al. (2006), Lockwood and Linn (1990), Bollerslev et al. (2000), and Andersen and Bollerslev (1997)). For the lower frequency of these variables, calendar effects are well documented in the literature (Hansen and Lunde (2005), Jordan and Jordan (1991), Andersen and Bollerslev (1998), and Lucia and Schwartz (2002)). Phillips (2001) and Franses (1996) discuss importance of direct modeling of trends and seasonalities, respectively. Model (1) has a potential to capture these time-dependent properties of the underlying process.

Time-inhomogeneous models suggested in the literature include Ho and Lee (1986), Hull and White (1990, 1994), and Black and Karasinski (1991) where they adopt time-inhomogeneous diffusion model for the short-term interest rate to match the current yield curve and volatility structure when pricing bond and bond options. In addition, Fan et al. (2003) use nonparametric techniques to estimate time-inhomogeneous

diffusion models for the 3 month T-bill rates and the results support their time-inhomogeneous diffusion. Time-inhomogeneous affine processes have been studied by Filipović (2005). Stock return volatility is dependent on both stock price and time in Rubinstein (1994) and Derman and Kani (1994).

The problem with regard to estimation of diffusion models is that the observed data are discrete while the model specification is in continuous-time. Ignoring the difference is known to result in inconsistent estimators (Merton (1980), Lo (1988) and Melino (1994)). In recent years, a lot of econometric methods have been proposed to estimate continuous-time diffusion models without requiring a continuous record of observations, for example, maximum likelihood estimation (MLE) (Pearson and Sun (1994), Chen and Scott (1993), Pedersen (1995), Santa-Clara (1995) and Durham and Gallant (2002)), simulation method (Gouriéroux et al. (1993), Duffie and Singleton (1993)), generalized method of moment (GMM) (Hansen and Scheinkman (1995), Duffie and Glynn (2004), Bibby and Sørensen (1995) and Kessler and Sørensen (1999)), efficient method of moment (EMM) (Gallant and Tauchen (1996,1998)), nonparametric method (Aït-Sahalia (1996a, 1996b), Stanton (1997) and Arapis and Gao (2006)) and Bayesian method (Eraker (2001), Elerian et al. (2001) and Jones (2003)).

MLE method has not been used much because the closed-form transition density function is unknown for most of diffusion processes. Therefore, people have numerically solved Kolmogorov partial differential equation which the transition density function should satisfy or simulated sample paths very finely along which the process is sampled in order to approximate the transition density. Neither of these methods produces a closed-form density function. Aït-Sahalia (2002) has established a way to get an approximate transition density function explicitly using Hermite series expansion for univariate diffusion models. His work is extended to univariate models driven by Levy processes by Schaumburg (2001), to univariate time-inhomogeneous diffusion models by Egorov et al. (2003), to a Bayesian setting by DiPietro (2001), and to a damped diffusion by Li (2010) and Choi (2009) applies his method to regime-switching univariate diffusion models for the short-term interest rate. Closed-form log-likelihood expansions for multivariate diffusions also have been obtained by Aït-Sahalia (2008), which is generalized to multivariate jump diffusions by Yu (2007) and Purzitsky (2003). This method has been applied to stochastic volatility and affine term structure models (Aït-Sahalia and Kimmel (2007, 2010) and Egorov, Li, and Ng (2011)). This article is built on Aït-Sahalia (2008) to extend his results to time-inhomogeneous cases.

The purpose of this paper is to approximate true but in most of cases unknown transition log-likelihood function of multivariate time-inhomogeneous diffusion processes in an explicit form. We need mild technical assumptions such as smoothness and linear growth conditions for the drift and volatility functions, for the existence and uniqueness of the solution to SDE (2) and to get a closed-form expansion of the likelihood.

A diffusion process is called reducible if there exists a transformation which can transform the original process into a unit diffusion whose volatility is an identity matrix. Otherwise it is said to be irreducible. If diffusions are reducible, two reducible methods (Hermite-expansion and Kolmogorov-equation methods) can

be used to obtain the approximate log-transition density function. In fact, most of multivariate diffusions are irreducible. So it is more important to find a way to get an approximate log-transition density for the irreducible case. When a multivariate time-homogeneous diffusion is irreducible we can no longer apply any of the reducible methods. The key idea of finding approximate log-transition densities of irreducible diffusions is to postulate the form of the log-likelihood expansion as the one attained from the reducible case and then to use the Kolmogorov equations to get the partial differential equations (PDEs) of each coefficient. Unlike the reducible case, generally these PDEs cannot be solved analytically. Aït-Sahalia (2008) Taylor expands each coefficient of the postulated log-transition density function and matches the same order terms in each PDE to find the coefficients of the Taylor-expansions. This must be done successively from low to high order of the Taylor-expansion and from low to high order of the coefficients of the log-transition density because high order terms depend on low order terms.

In contrast to the time-homogeneous case, it can be shown that infinitely many terms of the Taylor-expansions of the first two coefficients of the log-likelihood expansion for a time-inhomogeneous diffusion cannot be determined from the PDEs. Therefore the recursive way of getting the coefficients breaks down. However, we prove that the approximate log-likelihood expansion itself does not depend on those indeterminate terms because they are cancelled out in the log-likelihood expansion although most of the coefficients of Taylor-expansions are nonzero in general and suffer from indeterminacy problems. This is the major contribution of this article to the literature. Consequently, we can safely set the indeterminate terms to zero while obtaining the coefficients of Taylor-expansions in a recursive way to construct approximate log-transition density expansions of time-inhomogeneous diffusion processes.

Egorov et al. (2003) employed the Hermite-expansion method to find an approximate transition density function for a univariate time-inhomogeneous diffusion. This paper provides an alternative form of the approximate transition density function to Egorov et al. (2003) by applying the Kolmogorov-equation method. As explained in Section 3.1, if we cannot find an antiderivative for the transformation of a reducible diffusion, Kolmogorov-equation method cannot be used regardless of whether it is time-homogeneous or time-inhomogeneous. Even so, Hermite-expansion method is still applicable to time-homogeneous reducible diffusions. However, when there is no explicit form for the derivatives of the transformation with respect to time variable either, none of the reducible methods can be used for time-inhomogeneous reducible diffusion models. So, Egorov et al. (2003)'s approach is not applicable to such cases. Fortunately, the irreducible method introduced in Section 3.2.2 can resolve this problem, too.

We conduct Monte Carlo simulation studies to examine performance of our methods when the approximate transition density function is used to get ML estimates. Both reducible and irreducible methods are applied to a benchmark model, bivariate time-inhomogeneous Ornstein-Uhlenbeck process whose true transition density function is known. And then the irreducible method is adopted for time-inhomogeneous versions of Heston's stochastic volatility and BDFS models to see the effectiveness of our new approach.

We have found that biases and standard errors of ML estimators calculated from using our approximate transition density are small and comparable to those obtained from maximizing the true transition density. These results provide strong evidences that both the reducible and irreducible methods yield an accurate approximation to the true density function of a time-inhomogeneous diffusion and they can be employed when the true density is unknown as is often the case.

Next section introduces our time-inhomogeneous diffusion models and states required assumptions in more detail. Section 3 describes main results of how to find an approximate log-transition density function for both reducible and irreducible cases. Convergence of the log-likelihood function and its maximizer is discussed in Section 4. Monte Carlo simulation results are given in Section 5. Then conclusion follows.

2 Model and Assumptions

Consider an m -dimensional multivariate time-inhomogeneous diffusion process,

$$dX_t = \mu(t, X_t; \theta) dt + \sigma(t, X_t; \theta) dW_t, \quad (2)$$

where X_t is an $m \times 1$ vector of state variables in $S_X \subset R^m$. The drift, $\mu(t, X_t; \theta)$ is an $m \times 1$ vector of functions, which are also known as infinitesimal mean or expected infinitesimal displacement, and $\sigma(t, X_t; \theta)$ is an $m \times m$ volatility (or dispersion) matrix. In our model, $\mu(t, X_t; \theta)$ or $\sigma(t, X_t; \theta)$ depend not only on the state variables X_t but also on time variable t and they are known up to a parameter vector $\theta \in \Theta$, which is a compact subset of R^p . For notational simplicity, we will omit the parameter vector θ until Section 4 where we discuss maximum likelihood estimator of θ . Without loss of generality, W_t is an $m \times 1$ vector of independent Brownian motions. Any correlation structures between the state variables can be modelled by using off-diagonal terms in the dispersion matrix, which needs not be symmetric. Let $S_X \subset R^m$ be the domain of the diffusion X_t . For simplicity, we assume the following for S_X .

Assumption 1. S_X is a product of m intervals with lower and upper limits, \underline{x}_i and \bar{x}_i , where possibly $\underline{x}_i = -\infty$ and/or $\bar{x}_i = \infty$, in which case the intervals are open at infinite limits.

Characterization of a diffusion depends on the following infinitesimal variance-covariance matrix, $v(t, x)$ rather than the dispersion matrix, $\sigma(t, x)$.

$$v(t, x) \equiv \sigma(t, x) \sigma^T(t, x), \quad (3)$$

where $\sigma^T(t, x)$ is transposition of $\sigma(t, x)$.

Assumption 2. The diffusion matrix $v(t, x)$ is positive definite for all (t, x) in the interior of $[0, \infty) \times S_X$.

Even though there may be a continuum of choices for $\sigma(t, x)$ satisfying (3), the transition probability of the process is the same for each one of these $\sigma(t, x)$ (see Remark 5.17 and Section 5.3 in Stroock and Varadhan (1979)). If $\mu(t, x) = \mu'(t, x)$ and $v(t, x) = v'(t, x)$ for two processes X_t and X'_t , two have the

same joint distribution function given that X_0 and X'_0 have the identical distribution (Friedman (1975) Vol 1. p151).

Assumption 3. For $i, j = 1, 2, \dots, m$, $\mu_i(t, x)$ and $\sigma_{ij}(t, x)$ are infinitely differentiable with respect to $x \in S_X$ and $t \in [0, \infty)$.

This infinite differentiability of drift and dispersion coefficients is necessary for computing the approximate expansions of the log-transition density. As a by-product of Assumption 3, the uniqueness of solution to the SDE (2) is also guaranteed. In fact, we only need local Lipschitz continuity of $\mu(t, x)$ and $\sigma(t, x)$ in x ; i.e., for every $C > 0$ there exists a constant $K_C > 0$ such that for every $t \in [0, \infty)$, $\|x\| \leq C$ and $\|x'\| \leq C$:

$$\|\mu(t, x) - \mu(t, x')\| + \|\sigma(t, x) - \sigma(t, x')\| \leq K_C \|x - x'\|$$

in order to ensure the uniqueness of solution to (2) if it exists (see e.g. Theorem 5.2.5 in Karatzas and Shreve (1998)). The notation $\|\cdot\|$ stands for the usual Euclidean norm. The local Lipschitz continuity can be obtained from once differentiability of the drift and volatility functions in Assumption 3 by applying the mean value theorem of calculus to $\mu_i(t, x)$ and $\sigma_{ij}(t, x)$.

In addition to the local Lipschitz condition, we need further restrictions on the coefficients, $\mu(t, x)$ and $\sigma(t, x)$ for the existence of a non-exploding solution to the SDE (2).

Assumption 4. There is a constant $K > 0$ such that for all $(t, x) \in [0, \infty) \times S_X$

$$\|\mu(t, x)\|^2 \leq K(1 + \|x\|^2) \text{ and } \|\sigma(t, x)\|^2 \leq K(1 + \|x\|^2) \quad (4)$$

Derivatives of $\mu_i(t, x)$'s and $\sigma_{ij}(t, x)$'s exhibit at most polynomial growth.

Assumption 4 ensures the existence of a non-exploding solution by keeping the rate of growth of both $\mu(t, x)$ and $\sigma(t, x)$ at most linear in x as $\|x\|$ goes to infinity. The polynomial growth assumption for the derivatives of the drift and dispersion functions simplifies things in view of the fact that the tails of the transition density function decrease exponentially. Assumption 4 can be relaxed to

$$\sum_{i=1}^m x_i \mu_i(t, x) \leq K(1 + \|x\|^2)$$

with the same restriction for $\sigma(t, x)$ to cover, for instance $\mu(t, x) = x - x^3$ case.

The Lipschitz condition plays a key role in the proof of the uniqueness of the solution. But, some multivariate diffusion models, for example the extended multivariate Cox-Ingersoll-Ross model with $\sigma(t, x) = \text{diag}(\sigma_{0i} \sqrt{x_i} e^{\sigma_{1i} t})_{i=1,2,\dots,m}$, does not meet the Lipschitz condition, not to mention Assumption 3. Here $\text{diag}(a_i)_{i=1,2,\dots,m}$ is the $m \times m$ diagonal matrix whose diagonal elements are a_i , $i = 1, 2, \dots, m$. Note that x^α is not differentiable at $x = 0$ and does not satisfy a local Lipschitz condition in a neighborhood of $x = 0$ for $0 < \alpha < 1$. Assumption 3 can be relaxed to insure the uniqueness (see Watanabe and Yamada (1971a, b) for more details). Since their work, Sonoc (1998) and Swart (2001) have refined the uniqueness conditions.

3 Closed-Form Log-Transition Density Expansions

3.1 Univariate Time-Inhomogeneous Diffusions

Let's first review the results for univariate time-inhomogeneous diffusions by Egorov et al. (2003) and discuss an alternative approach to find approximate transition density functions and a problem in these two approaches. We need to transform the original diffusion process X_t twice in order to make the transition density of the transformed diffusion process close to the Normal. The transition density function of X_t can be recovered using Jacobian.

First, transform univariate diffusion process, (2) to a unit diffusion process, (6) by the following Lamperti transformation from X to Y ,

$$y = \gamma(t, x) \equiv \int^x \frac{1}{\sigma(t, \omega)} d\omega. \quad (5)$$

Using Ito's formula,

$$dY_t = \mu_Y(t, Y_t) dt + dW_t, \quad (6)$$

and $\mu_Y(t, y) = \frac{\mu[\gamma^{inv}(t, y), t]}{\sigma[\gamma^{inv}(t, y), t]} - \frac{1}{2} \frac{\partial \sigma[\gamma^{inv}(t, y), t]}{\partial x} + \frac{\partial \gamma[\gamma^{inv}(t, y), t]}{\partial t}$, where $\gamma^{inv}(t, y)$ denotes the inverse of a transformation γ from $x \in R^m$ to $y \in R^m$, $y = \gamma(t, x)$ when it exists. The first transformation makes the tail behavior of p_Y close to Gaussian by making the dispersion term equal to 1. However, p_Y gets concentrated and peaked around the conditional value y_0 when the time difference between two consecutive observations, $\Delta = t - t_0$, becomes small. Hence, we need the second transformation from Y to Z

$$z \equiv \frac{y - y_0}{\sqrt{\Delta}}.$$

This transformation makes the transition density of Z_t , p_Z close to the Normal. As a consequence, Hermite-expansion of p_Z around standard normal density function, $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$,

$$p_Z^{(J)}(t, z | t_0, y_0) \equiv \phi(z) \sum_{j=0}^J \eta_Z^{(j)}(t, t_0, y_0) H_j(z) \quad (7)$$

can produce an accurate approximate density with relatively a few coefficients. Hermite polynomials $H_j(z)$'s are defined by $H_j(z) \equiv \phi(z)^{-1} \frac{d^j}{dz^j} \phi(z)$, for example $H_0(z) = 1$, $H_1(z) = -z$, $H_2(z) = z^2 - 1$, $H_3(z) = -z^3 + 3z$, $H_4(z) = z^4 - 6z^2 + 3$, $H_5(z) = -z^5 + 10z^3 - 15z$ and $H_6(z) = z^6 - 15z^4 + 45z^2 - 15$ and these are orthogonal base functions since

$$\int_{-\infty}^{\infty} H_i(z) H_j(z) \phi(z) dz = \begin{cases} i! & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Using the orthogonality of Hermite polynomials we can obtain the coefficients $\eta^{(j)}(\Delta, y_0, s)$ of the Hermite

expansion (7) such that

$$\begin{aligned}
\eta_Z^{(j)}(t, t_0, y_0) &= (1/j!) \int_{-\infty}^{\infty} H_j(z) p_Z(t, z|t_0, y_0) dz \\
&= (1/j!) \int_{-\infty}^{\infty} H_j(z) \sqrt{\Delta} p_Y\left(t, \sqrt{\Delta}z + y_0|t_0, y_0\right) dz \\
&= (1/j!) \int_{-\infty}^{\infty} H_j\left(\frac{y - y_0}{\sqrt{\Delta}}\right) p_Y(t, y|t_0, y_0) dy \\
&= (1/j!) E\left[H_j\left(\frac{Y_t - y_0}{\sqrt{\Delta}}\right) \middle| Y_{t_0} = y_0\right].
\end{aligned}$$

The conditional expectation can be approximated up to any order by Taylor expansion using infinitesimal operator,¹ A_Y defined by

$$A_Y \circ f(t, y, t_0, y_0) = \frac{\partial f(t, y, t_0, y_0)}{\partial t} + \mu_Y(t, y) \frac{\partial f(t, y, t_0, y_0)}{\partial y} + \frac{1}{2} \frac{\partial^2 f(t, y, t_0, y_0)}{\partial y^2}.$$

Therefore, for any infinitely differentiable function f

$$E[f(t, Y_t, t_0, Y_{t_0}) | Y_{t_0} = y_0] = \sum_{i=0}^K \frac{\Delta^i}{i!} A_Y^i \circ f(t, y, t_0, y_0) \Big|_{y=y_0, t=t_0} + O(\Delta^{K+1})$$

In this way, Egorov et al. (2003) approximate p_Y as

$$p_Y^{(J,K)}(t, y|t_0, y_0) = \Delta^{-\frac{1}{2}} \phi\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \left\{ \sum_{j=0}^J \frac{1}{j!} \left[\sum_{i=0}^K \frac{\Delta^i}{i!} A_Y^i \circ H_j\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \Big|_{y=y_0} \right] H_j\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \right\} \quad (8)$$

That is to say, we have to calculate $\eta_Z^{(j)}(t, y_0|t_0)$'s up to K -th order and truncate (7) at J in order to find the approximation, $p_Y^{(J,K)}$. We will call this Hermite-expansion method.

There is an alternative way to obtain the approximate expansion to p_Y . After letting $J \rightarrow \infty$ in $p_Y^{(J,K)}$ if we rearrange all terms in (8) according to powers of Δ , the resulting expansion up to K -th order is given by (9) and its coefficients can be calculated recursively by (10).

$$p_Y^{(K)}(t, y|t_0, y_0) = \Delta^{-\frac{1}{2}} \phi\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \exp\left(\int_{y_0}^y \mu_Y(t, \omega) d\omega\right) \sum_{k=0}^K c_Y^{(k)}(t, y|t_0, y_0) \frac{\Delta^k}{k!}, \quad (9)$$

¹In general, if a univariate diffusion process X_t follows

$$X_t = \mu(X_t, t; \theta) dt + \sigma(X_t, t; \theta) dW_t$$

then

$$\begin{aligned}
\lim_{\Delta \rightarrow 0} \frac{U_{\Delta} g(X_t, t) - U_0 g(X_t, t)}{\Delta} &= \lim_{\Delta \rightarrow 0} \frac{E[g(X_t, t) | X_s = x_0] - g(x_0, s)}{\Delta} = A_X \circ g(x, t) \\
&= \frac{\partial g(x, t)}{\partial t} + \mu(x, t; \theta) \frac{\partial g(x, t)}{\partial x} + \frac{1}{2} \sigma^2(x, t; \theta) \frac{\partial^2 g(x, t)}{\partial x^2} \Big|_{x=x_0, t=s}
\end{aligned}$$

where $c_Y^{(0)}(t, y|t_0, y_0) = 1$ and for all $k \geq 1$

$$c_Y^{(k)}(t, y|t_0, y_0) = k \int_0^1 u^{k-1} \left\{ \lambda_Y(t, y_0 + u(y - y_0), y_0) c_Y^{(k-1)}(t, y_0 + u(y - y_0)|t_0, y_0) \right. \\ \left. + \frac{1}{2} \frac{\partial^2 c_Y^{(k-1)}(t, \omega|t_0, y_0)}{\partial \omega^2} \Big|_{\omega=y_0+u(y-y_0)} - \frac{\partial c_Y^{(k-1)}(t, y_0 + u(y - y_0)|t_0, y_0)}{\partial t} \right\} du, \quad (10)$$

where

$$\lambda_Y(t, y, y_0) = -\frac{1}{2} \left[\mu_Y(t, y)^2 + \frac{\partial \mu_Y(t, y)}{\partial y} \right] - \int_{y_0}^y \frac{\partial \mu_Y(t, \omega)}{\partial t} d\omega.$$

For the case of univariate time-homogeneous model, Aït-Sahalia (1999) verifies that we can get very good approximate expansion with $K = 1$ or $K = 2$ at most.

The closed-form expansion of the logarithm of the transition density function can be found by Taylor-expanding $\ln \left(\sum_{k=0}^K c_Y^{(k)}(t, y|t_0, y_0) \frac{\Delta^k}{k!} \right)$ obtained from $\ln [p_Y^{(K)}(t, y|t_0, y_0)]$ in Δ rather than just taking logarithm of (9). The K -th order approximate log-likelihood function of (6) is given by

$$l_Y^{(K)}(t, y | t_0, y_0) = -\frac{1}{2} \ln(2\pi\Delta) + \frac{C_Y^{(-1)}(t, y | t_0, y_0)}{\Delta} + \sum_{k=0}^K C_Y^{(k)}(t, y | t_0, y_0) \frac{\Delta^k}{k!} \quad (11)$$

and the coefficients are

$$C_Y^{(-1)}(t, y | t_0, y_0) = -(y - y_0)^2 / 2 \\ C_Y^{(0)}(t, y | t_0, y_0) = (y - y_0) \int_0^1 \mu_Y(t, y_0 + u(y - y_0)) du.$$

Given $C_Y^{(-1)}, C_Y^{(0)}, \dots, C_Y^{(k-1)}, C_Y^{(k)}$, $k \geq 1$, can be calculated recursively by

$$C_Y^{(k)}(t, y | t_0, y_0) = k \int_0^1 G_Y^{(k)}(t, y_0 + u(y - y_0) | t_0, y_0) u^{k-1} du$$

where

$$G_Y^{(1)}(t, y | t_0, y_0) = \lambda_Y(t, y, y_0)$$

and for $k \geq 2$

$$G_Y^{(k)}(t, y | t_0, y_0) = \frac{1}{2} \frac{\partial^2 C_Y^{(k-1)}(t, y | t_0, y_0)}{\partial y^2} - \frac{\partial C_Y^{(k-1)}(t, y | t_0, y_0)}{\partial t} \\ + \frac{1}{2} \sum_{h=1}^{k-2} \binom{k-1}{h} \frac{\partial C_Y^{(h)}(t, y | t_0, y_0)}{\partial y} \frac{\partial C_Y^{(k-1-h)}(t, y | t_0, y_0)}{\partial y} \quad (12)$$

The binomial coefficient is written as $\binom{k}{j} \equiv \frac{k!}{j!(k-j)!}$ in the paper. This way of expanding log-transition function allows us to calculate the log-likelihood more easily. Taking exponential of the log-likelihood expansion yields an approximate transition density function that is positive. In Section 3.2.1, we will prove these results in a more general setting of an m -dimensional diffusion case. In fact, Kolmogorov equations play a key role in finding the coefficients of (9) and (11). So, let us call this Kolmogorov-equation method.

It is obvious that Kolmogorov-equation method cannot be applied to any of univariate diffusions regardless of whether it is time-homogeneous or time-inhomogeneous when the integration (5) cannot be done explicitly, for example $\sigma(t, x) = (\theta_{-1}x^{-1} + \theta_0 + \theta_1x + \theta_2x^{\theta_3}e^{\theta_4t})$, where θ_i 's are parameters. However, we can still derive the approximate density function for time-homogeneous univariate diffusions whether or not there exists an explicit formula for $\gamma(x)$ as long as we use the Hermite-expansion method. To see this, remember that what we need is the approximate transition density function of the diffusion X_t not Y_t and it can be recovered by the change of variable:

$$p_X^{(J,K)}(\Delta, x|x_0) = \frac{p_Y^{(J,K)}[\Delta, \gamma(x)|\gamma(x_0)]}{\sigma(x)}.$$

Hence the coefficients of the expansion, $p_X^{(J,K)}(\Delta, x|x_0)$ are $\eta_Z^{(j)}(\Delta, \gamma(x_0))$. Notice that the coefficients, $\eta_Z^{(j)}(\Delta, y_0)$'s in the Hermite-expansion of $p_Y^{(J,K)}(\Delta, y|y_0)$ are approximated as a function of $\mu_Y(y)$ and its differentiations $d^k \mu_Y(y)/dy^k|_{y=y_0}$, $k \geq 1$ evaluated at $y = y_0$, where $\mu_Y(y) = \frac{\mu[\gamma^{inv}(y)]}{\sigma[\gamma^{inv}(y)]} - \frac{1}{2} \frac{\partial \sigma(x)}{\partial x} \Big|_{x=\gamma^{inv}(y)}$ (Aït-Sahalia (2002)). Because $\frac{\mu(w)}{\sigma(w)} - \frac{1}{2} \frac{\partial \sigma(w)}{\partial w}$ is a known function and $d\gamma^{inv}(y)/dy = 1 / \left[d\gamma(x)/dx \Big|_{x=\gamma^{inv}(y)} \right] = \sigma(\gamma^{inv}(y))$, $\eta_Z^{(j)}(\Delta, \gamma(x_0))$'s of $p_X^{(J,K)}(\Delta, x|x_0)$ depend on μ , σ and their derivatives at $x = x_0$. Therefore $p_X^{(J,K)}(\Delta, x|x_0)$ is obtainable even though it is impossible to find an explicit formula for $\gamma(x)$. Unfortunately, this is not possible in the case of univariate time-inhomogeneous diffusion unless there is an explicit form of $\partial^k \gamma(t, x)/\partial t^k$, $k \geq 0$, for instance, $\sigma(t, x) = (\theta_{-1}x^{-1} + \theta_0 + \theta_1x + \theta_2x^{\theta_3}e^{\theta_4t})$. So, Egorov et al. (2003)'s approach is not applicable to such a case. In order to show this, note that the coefficients, $\eta_Z^{(j)}(t, t_0, y_0)$'s of $p_Y^{(J,K)}(t, y|t_0, y_0)$ are dependent on $\mu_Y(t, y) = \frac{\mu[\gamma^{inv}(t, y), t]}{\sigma[\gamma^{inv}(t, y), t]} - \frac{1}{2} \frac{\partial \sigma[\gamma^{inv}(t, y), t]}{\partial x} + \frac{\partial \gamma[\gamma^{inv}(t, y), t]}{\partial t}$ and its differentials with respect to t and y at $t = t_0$ and $y = y_0$. Therefore, generally $p_X^{(J,K)}(t, x|t_0, x_0)$ cannot be found without knowing the explicit formula of $\partial \gamma(t, x)/\partial t$ since we must be able to compute

$$\frac{\partial \gamma^{inv}(t, y)}{\partial t} \Big|_{t=t_0, y=y_0} = - \frac{\partial \gamma(t, x)/\partial t}{\sigma(\gamma^{inv}(t, y), t)} \Big|_{t=t_0, y=\gamma(t_0, x_0)},$$

and higher order derivatives of $\gamma(t, x)$ with respect to t to get $\eta_Z^{(j)}(t, t_0, \gamma(t_0, x_0))$. As can be seen below, the same problem arises in the multivariate reducible case. Fortunately, this problem is resolved if we use the irreducible method discussed in section 3.2.2.

3.2 Multivariate Time-Inhomogeneous Diffusions

First, we need to introduce reducibility of a diffusion process. If a multivariate diffusion process is reducible, we can use the Hermite-expansion or Kolmogorov-equation methods as we have done for a univariate process to find its log-likelihood expansion. On the other hand, if a multivariate diffusion is not reducible (or irreducible) we can no longer turn to any of these approaches. As we will see in section 3.2.2, the irreducible method enables us to find the approximate log-likelihood function of X_t without transforming it to a unit diffusion.

Definition : (Reducibility) The diffusion process X_t is said to be reducible to unit diffusion (or reducible, in short) if and only if there exists a one-to-one transformation of the diffusion process X_t into a diffusion process Y_t whose dispersion matrix σ_Y is the identity matrix. That is, there exists an invertible function $\gamma(t, x)$, infinitely differentiable in t and x on $[0, \infty) \times S_X$ such that $Y_t \equiv \gamma(t, X_t)$ satisfies the stochastic differential equation

$$dY_t = \mu_Y(t, Y_t) dt + dW_t$$

on the domain S_Y . The drift function of the i -th component of Y_t is

$$\begin{aligned} \mu_{Y_i}(t, y) &= \frac{\partial \gamma_i(t, \gamma^{inv}(t, y))}{\partial t} + \sum_{p=1}^m \mu_p(t, \gamma^{inv}(t, y)) \frac{\partial \gamma_i(t, x)}{\partial x_p} \Big|_{x=\gamma^{inv}(t, y)} \\ &+ \frac{1}{2} \sum_{p=1}^m \sum_{q=1}^m \sum_{r=1}^m \sigma_{pr}(t, \gamma^{inv}(t, y)) \sigma_{qr}(t, \gamma^{inv}(t, y)) \frac{\partial^2 \gamma_i(t, x)}{\partial x_p \partial x_q} \Big|_{x=\gamma^{inv}(t, y)} \end{aligned}$$

and

$$\nabla \gamma(t, x) = \sigma^{-1}(t, x) \quad (13)$$

by applying the Ito's lemma.

In the above definition, $\nabla \gamma(t, x)$ represents the Jacobian matrix of $\gamma(t, x)$ with respect to $x \in R^m$ such that $\nabla \gamma(t, x) = [\partial \gamma_i(t, x) / \partial x_j]_{i=1, \dots, m; j=1, \dots, m}$. Every univariate diffusion process can be transformed into a unit diffusion by (5). But, there doesn't always exist the transformation $\gamma(t, x)$ for a multivariate diffusion. We can tell whether a diffusion process is reducible or not by checking the conditions given below.

Proposition 1 (Necessary and sufficient condition for reducibility) The diffusion process X_t is reducible if and only if

$$\sum_{l=1}^m \frac{\partial \sigma_{ik}(t, x)}{\partial x_l} \sigma_{lj}(t, x) = \sum_{l=1}^m \frac{\partial \sigma_{ij}(t, x)}{\partial x_l} \sigma_{lk}(t, x)$$

for each x in S_X and triplet $\{i, j, k\} \subset \{1, 2, \dots, m\}$ such that $k > j$. If σ is nonsingular, then the condition can be expressed as

$$\frac{\partial \sigma_{ij}^{-1}(t, x)}{\partial x_k} = \frac{\partial \sigma_{ik}^{-1}(t, x)}{\partial x_j},$$

where $\sigma_{ij}^{-1}(t, x)$ is the (i, j) element of the inverse matrix of $\sigma(t, x)$.

Reducibility conditions have to do with only the dispersion matrix $\sigma(t, x)$. According to this proposition, when σ is nonsingular, $m^2(m-1)/2$ equalities must hold in order for an m -dimensional diffusion to be reducible. For example, when $m = 2$ we need to check only two equalities.

$$\frac{\partial \sigma_{11}^{-1}(t, x)}{\partial x_2} = \frac{\partial \sigma_{12}^{-1}(t, x)}{\partial x_1} \text{ and } \frac{\partial \sigma_{22}^{-1}(t, x)}{\partial x_1} = \frac{\partial \sigma_{21}^{-1}(t, x)}{\partial x_2} \quad (14)$$

If $m = 3$, in addition to (14), we need to look into seven more equalities, which are

$$\frac{\partial \sigma_{11}^{-1}(t, x)}{\partial x_3} = \frac{\partial \sigma_{13}^{-1}(t, x)}{\partial x_1}, \frac{\partial \sigma_{22}^{-1}(t, x)}{\partial x_3} = \frac{\partial \sigma_{23}^{-1}(t, x)}{\partial x_2}, \frac{\partial \sigma_{33}^{-1}(t, x)}{\partial x_1} = \frac{\partial \sigma_{31}^{-1}(t, x)}{\partial x_3},$$

$$\frac{\partial \sigma_{33}^{-1}(t, x)}{\partial x_2} = \frac{\partial \sigma_{32}^{-1}(t, x)}{\partial x_3}, \frac{\partial \sigma_{21}^{-1}(t, x)}{\partial x_3} = \frac{\partial \sigma_{23}^{-1}(t, x)}{\partial x_1},$$

$$\frac{\partial \sigma_{12}^{-1}(t, x)}{\partial x_3} = \frac{\partial \sigma_{13}^{-1}(t, x)}{\partial x_2} \text{ and } \frac{\partial \sigma_{31}^{-1}(t, x)}{\partial x_2} = \frac{\partial \sigma_{32}^{-1}(t, x)}{\partial x_1}$$

3.2.1 Reducible Diffusions

To construct approximate log-transition density function of a multivariate time-inhomogeneous reducible diffusion process we can follow the same line as the univariate case. Let us first discuss Hermite-expansion approach. Multivariate Hermite polynomials are defined by

$$H_h(z) = \frac{(-1)^{|h|}}{\phi(z)} \frac{\partial^{|h|} \phi(z)}{\partial z_1^{h_1} \dots \partial z_m^{h_m}},$$

where h is an m -dimensional vector with non-negative integers h_i 's and $\phi(z)$ is the density of the m -dimensional multivariate Normal distribution with mean zero and identity covariance matrix. In order to represent the sum of the all elements of an m dimensional vector h , $|h|$ will be used, i.e., $|h| = \sum_{i=1}^m h_i$. The Hermite polynomials can be computed explicitly to any order $h = (h_1, \dots, h_m)$ (see Chapter 5 of McCullagh (1987) or Withers (2000)). The Hermite polynomials are orthogonal in the following sense

$$\int_{R^m} H_h(z) H_k(z) \phi(z) dx = \begin{cases} h_1! \dots h_m! & \text{if } h = k \\ 0 & \text{otherwise} \end{cases}.$$

The J -th order Hermite series expansion, $p_Y^{(J)}$ for a multivariate time-inhomogeneous unit diffusion process Y_t is of the form

$$p_Y^{(J)}(t, y | t_0, y_0) = \Delta^{-m/2} \phi\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \sum_{h \in N^m: |h| \leq J} \eta_h(t, t_0, y_0) H_h\left(\frac{y - y_0}{\sqrt{\Delta}}\right)$$

Using the orthogonality of the Hermite polynomials,

$$\eta_h(t, t_0, y_0) = \frac{1}{h_1! \dots h_m!} E \left[H_h\left(\frac{Y_t - y_0}{\sqrt{\Delta}}\right) \middle| Y_{t_0} = y_0 \right].$$

If the coefficient, $\eta_h(t, t_0, y_0)$ is approximated up to K -th order by using the infinitesimal generator A_Y corresponding to the diffusion process Y_t , i.e.,

$$A_Y \circ f(t, y, t_0, y_0) = \frac{\partial f(t, y, t_0, y_0)}{\partial t} + \sum_{i=1}^m \mu_{Y_i}(t, y) \frac{\partial f(t, y, t_0, y_0)}{\partial y_i} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^2 f(t, y, t_0, y_0)}{\partial y_i \partial y_j}.$$

then, we attain

$$p_Y^{(J, K)}(t, y | t_0, y_0) = \Delta^{-m/2} \phi\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \left\{ \sum_{h \in N^m: |h| \leq J} \frac{1}{h_1! \dots h_m!} \left[\sum_{i=0}^K \frac{\Delta^i}{i!} A_Y^i \circ H_h\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \middle|_{y=y_0} \right] H_h\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \right\}.$$

We can send J to ∞ and rearrange the terms of $p_Y^{(J,K)}$ in ascending orders of Δ to acquire an alternative expansion $p_Y^{(K)} = p_Y^{(\infty,K)}$ such that

$$p_Y^{(K)}(t, y | t_0, y_0) = \Delta^{-m/2} \phi\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \exp\left[\sum_{i=1}^m (y_i - y_{0i}) \int_0^1 \mu_{Y_i}(t, y_0 + u(y - y_0)) du\right] \sum_{k=0}^K c_Y^{(k)}(t, y | t_0, y_0) \frac{\Delta^k}{k!}. \quad (15)$$

Taking logarithm of (15) and Taylor-expanding it in Δ around zero we get the approximate log-transition density of Y_t in Theorem 1 as a result of applying the Kolmogorov-equation method.

Theorem 1 *The K -th order log-transition density expansion of a time-inhomogeneous multivariate unit diffusion process Y_t is*

$$l_Y^{(K)}(t, y | t_0, y_0) = -\frac{m}{2} \ln(2\pi\Delta) + \frac{C_Y^{(-1)}(t, y | t_0, y_0)}{\Delta} + \sum_{k=0}^K C_Y^{(k)}(t, y | t_0, y_0) \frac{\Delta^k}{k!} \quad (16)$$

with

$$C_Y^{(-1)}(t, y | t_0, y_0) = -\frac{1}{2} \sum_{i=1}^m (y_i - y_{0i})^2 \quad (17)$$

$$C_Y^{(0)}(t, y | t_0, y_0) = \sum_{i=1}^m (y_i - y_{0i}) \int_0^1 \mu_{Y_i}(t, y_0 + u(y - y_0)) du \quad (18)$$

and for $k \geq 1$

$$C_Y^{(k)}(t, y | t_0, y_0) = k \int_0^1 G_Y^{(k)}(t, y_0 + u(y - y_0) | t_0, y_0) u^{k-1} du \quad (19)$$

where

$$\begin{aligned} G_Y^{(1)}(t, y | t_0, y_0) &= -\sum_{i=1}^m \frac{\partial \mu_{Y_i}(t, y)}{\partial y_i} - \frac{\partial C_Y^{(0)}(t, y | t_0, y_0)}{\partial t} \\ &\quad - \sum_{i=1}^m \mu_{Y_i}(t, y) \frac{\partial C_Y^{(0)}(t, y | t_0, y_0)}{\partial y_i} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \left\{ \frac{\partial^2 C_Y^{(0)}(t, y | t_0, y_0)}{\partial y_i^2} + \left[\frac{\partial C_Y^{(0)}(t, y | t_0, y_0)}{\partial y_i} \right]^2 \right\} \end{aligned} \quad (20)$$

and for $k \geq 2$

$$\begin{aligned} G_Y^{(k)}(t, y | t_0, y_0) &= \frac{1}{2} \sum_{i=1}^m \frac{\partial^2 C_Y^{(k-1)}(t, y | t_0, y_0)}{\partial y_i^2} - \frac{\partial C_Y^{(k-1)}(t, y | t_0, y_0)}{\partial t} \\ &\quad - \sum_{i=1}^m \mu_{Y_i}(t, y) \frac{\partial C_Y^{(k-1)}(t, y | t_0, y_0)}{\partial y_i} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \sum_{h=0}^{k-1} \binom{k-1}{h} \frac{\partial C_Y^{(h)}(t, y | t_0, y_0)}{\partial y_i} \frac{\partial C_Y^{(k-1-h)}(t, y | t_0, y_0)}{\partial y_i}. \end{aligned} \quad (21)$$

It is well known that the transition density $p_Y(t, y | t_0, y_0)$ satisfies the Kolmogorov forward and backward equations, respectively,

$$\frac{\partial p_Y(t, y | t_0, y_0)}{\partial t} = -\sum_{i=1}^m \frac{\partial \{\mu_{Y_i}(t, y) p_Y(t, y | t_0, y_0)\}}{\partial y_i} + \frac{1}{2} \sum_{i=1}^m \frac{\partial^2 p_Y(t, y | t_0, y_0)}{\partial y_i^2}$$

and

$$-\frac{\partial p_Y(t, y | t_0, y_0)}{\partial t_0} = \sum_{i=1}^m \mu_{Y_i}(t_0, y_0) \frac{\partial p_Y(t, y | t_0, y_0)}{\partial y_{0i}} + \frac{1}{2} \sum_{i=1}^m \frac{\partial^2 p_Y(t, y | t_0, y_0)}{\partial y_{0i}^2}.$$

If we plug the log-likelihood expansion (16) into the Kolmogorov forward equation for $l_Y(t, y | t_0, y_0)$:

$$\begin{aligned} \frac{\partial l_Y(t, y | t_0, y_0)}{\partial t} &= -\sum_{i=1}^m \frac{\partial \mu_{Y_i}(t, y)}{\partial y_i} - \sum_{i=1}^m \mu_{Y_i}(t, y) \frac{\partial l_Y(t, y | t_0, y_0)}{\partial y_i} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \frac{\partial^2 l_Y(t, y | t_0, y_0)}{\partial y_i^2} + \frac{1}{2} \sum_{i=1}^m \left[\frac{\partial l_Y(t, y | t_0, y_0)}{\partial y_i} \right]^2 \end{aligned} \quad (22)$$

and equate the coefficients of Δ^k , $k \geq -2$ on both sides of (22) then we get the PDEs for $C_Y^{(k)}(t, y | t_0, y_0)$, $k \geq -1$. Each of these PDEs can be solved as given in Theorem 1 by using Gaussianity of Y_t as $\Delta \rightarrow 0$, boundary conditions and the fact that for each term Δ^k , $k \geq 2$, (16) also satisfies the Kolmogorov backward equation for $l_Y(t, y | t_0, y_0)$:

$$\begin{aligned} -\frac{\partial l_Y(t, y | t_0, y_0)}{\partial t_0} &= \sum_{i=1}^m \mu_{Y_i}(t_0, y_0) \frac{\partial l_Y(t, y | t_0, y_0)}{\partial y_{0i}} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \frac{\partial^2 l_Y(t, y | t_0, y_0)}{\partial y_{0i}^2} + \frac{1}{2} \sum_{i=1}^m \left[\frac{\partial l_Y(t, y | t_0, y_0)}{\partial y_{0i}} \right]^2. \end{aligned} \quad (23)$$

The log-density, $l_X(t, x | t_0, x_0)$ of the original process X_t can be retrieved from $l_Y(t, y | t_0, y_0)$ by change of variable as

$$\begin{aligned} l_X(t, x | t_0, x_0) &= \ln(Det[\nabla \gamma(t, x)]) + l_Y(t, \gamma(t, x) | t_0, \gamma(t_0, x_0)) \\ &= -D_v(t, x) + l_Y(t, \gamma(t, x) | t_0, \gamma(t_0, x_0)). \end{aligned}$$

The second equality is because $Det[\nabla \gamma(t, x)] = Det[\sigma^{-1}(t, x)] = Det[v(t, x)]^{-1/2}$ and we define $D_v(t, x) \equiv \frac{1}{2} \ln(Det[v(t, x)])$. The determinant of a matrix a will be written as $Det[a]$. Replace l_Y with $l_Y^{(K)}$ found in Theorem 1 and define $l_X^{(K)}$ as

$$\begin{aligned} l_X^{(K)}(t, x | t_0, x_0) &= -D_v(t, x) + l_Y^{(K)}(t, \gamma(t, x) | t_0, \gamma(t_0, x_0)) \\ &= -D_v(t, x) - \frac{m}{2} \ln(2\pi\Delta) + \frac{C_Y^{(-1)}(t, \gamma(t, x) | t_0, \gamma(t_0, x_0))}{\Delta} \\ &\quad + \sum_{k=0}^K C_Y^{(k)}(t, \gamma(t, x) | t_0, \gamma(t_0, x_0)) \frac{\Delta^k}{k!}. \end{aligned} \quad (24)$$

By construction, $l_X^{(K)}$ solves the Kolmogorov equations for X_t at the same order as $l_Y^{(K)}$.

If $\mu_i(t, x) = \mu_i(t, x_i)$ and $\sigma_{ii}(t, x) = \sigma_{ii}(t, x_i)$ for all $i = 1, \dots, m$ and $\sigma_{ij}(t, x_i) = 0$ for $i \neq j$ then the transition density of the diffusion is just the product of the transition density of each variable. Reducibility of this special case is obvious and

$$l_X^{(K)}(t, x | t_0, x_0) = \sum_{i=1}^m l_{X_i}^{(K)}(t, x_i | t_0, x_{0i})$$

where $l_{X_i}^{(K)}(t, x_i | t_0, x_{0i})$ is the univariate expansion of the log-density for the i -th variable.

3.2.2 Irreducible Diffusions

As mentioned earlier, not every multivariate time-inhomogeneous diffusion is reducible. Moreover, the reducibility conditions in Proposition 1 are so restrictive that most of multivariate time-inhomogeneous diffusions are irreducible. Therefore it is much more important to find a way to get an approximate log-transition density for the irreducible case. If a multivariate time-inhomogeneous diffusion process, X_t is irreducible, none of reducible methods can be adopted since we cannot transform it into a unit diffusion process. Furthermore, as discussed in the univariate case, even if X_t is time-homogeneous and reducible, if there is no explicit formula for γ_i^{inv} , only the Hermite-expansion method is available. On the other hand, if it is time-inhomogeneous and there doesn't exist $\partial^k \gamma_i / \partial t^k$, $k \geq 1$, in closed form either, neither of the reducible methods can be applied but we can still turn to the irreducible method explained below.

The key idea of finding approximate log-transition density of irreducible diffusions is to postulate the form of the log-likelihood expansion of X_t as the one found from the reducible case:

$$l_X^{(K)}(t, x | t_0, x_0) = -\frac{m}{2} \ln(2\pi\Delta) - D_v(t, x) + \frac{C_X^{(-1)}(t, x | t_0, x_0)}{\Delta} + \sum_{k=0}^K C_X^{(k)}(t, x | t_0, x_0) \frac{\Delta^k}{k!} \quad (25)$$

Replacing l_X with $l_X^{(K)}$ in the Kolmogorov equations for the log-transition density of X_t and matching the terms with the same orders of Δ yields PDEs of the coefficients $C_X^{(k)}(t, x | t_0, x_0)$, $k \geq -1$ as given in Theorem 2. The Kolmogorov forward and backward equations for $l_X(t, x | t_0, x_0)$ are, respectively,

$$\begin{aligned} \frac{\partial l_X(t, x | t_0, x_0)}{\partial t} = & -\sum_{i=1}^m \frac{\partial \mu_i(t, x)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^2 v_{ij}(t, x)}{\partial x_i \partial x_j} - \sum_{i=1}^m \mu_i(t, x) \frac{\partial l_X(t, x | t_0, x_0)}{\partial x_i} \\ & + \sum_{i=1}^m \sum_{j=1}^m \frac{\partial v_{ij}(t, x)}{\partial x_i} \frac{\partial l_X(t, x | t_0, x_0)}{\partial x_j} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial^2 l_X(t, x | t_0, x_0)}{\partial x_i \partial x_j} \\ & + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial l_X(t, x | t_0, x_0)}{\partial x_i} v_{ij}(t, x) \frac{\partial l_X(t, x | t_0, x_0)}{\partial x_j} \end{aligned} \quad (26)$$

and

$$\begin{aligned} -\frac{\partial l_X(t, x | t_0, x_0)}{\partial t_0} = & \sum_{i=1}^m \mu_i(t_0, x_0) \frac{\partial l_X(t, x | t_0, x_0)}{\partial x_{0i}} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t_0, x_0) \frac{\partial^2 l_X(t, x | t_0, x_0)}{\partial x_{0i} \partial x_{0j}} \\ & + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial l_X(t, x | t_0, x_0)}{\partial x_{0i}} v_{ij}(t_0, x_0) \frac{\partial l_X(t, x | t_0, x_0)}{\partial x_{0j}}. \end{aligned} \quad (27)$$

Theorem 2 *The coefficients of the log-likelihood expansions of irreducible multivariate time-inhomogeneous diffusion processes, $C_X^{(k)}(t, x | t_0, x_0)$'s in (25) satisfy the following differential equations*

$$f_X^{(k-1)}(t, x | t_0, x_0) = 0$$

where $k = -1, 0, \dots, K$,

$$f_X^{(-2)}(t, x | t_0, x_0) = -2C_X^{(-1)}(t, x | t_0, x_0) - \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial C_X^{(-1)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial C_X^{(-1)}(t, x | t_0, x_0)}{\partial x_j}, \quad (28)$$

$$f_X^{(-1)}(t, x | t_0, x_0) = - \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial C_X^{(-1)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial C_X^{(0)}(t, x | t_0, x_0)}{\partial x_j} - G_X^{(0)}(t, x | t_0, x_0), \quad (29)$$

and for $k \geq 1$

$$\begin{aligned} f_X^{(k-1)}(t, x | t_0, x_0) &= C_X^{(k)}(t, x | t_0, x_0) \\ &\quad - \frac{1}{k} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial C_X^{(-1)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial C_X^{(k)}(t, x | t_0, x_0)}{\partial x_j} \\ &\quad - G_X^{(k)}(t, x | t_0, x_0). \end{aligned} \quad (30)$$

For $k = 0$ and 1 ,

$$G_X^{(0)}(t, x | t_0, x_0) = - \frac{\partial C_X^{(-1)}(t, x | t_0, x_0)}{\partial t} + G_X^{(0,1)}(t, x | t_0, x_0) + G_X^{(0,3)}(t, x | t_0, x_0),$$

where

$$\begin{aligned} G_X^{(0,1)}(t, x | t_0, x_0) &= - \sum_{i=1}^m \mu_i(t, x) \frac{\partial C_X^{(-1)}(t, x | t_0, x_0)}{\partial x_i} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m \frac{\partial v_{ij}(t, x)}{\partial x_i} \frac{\partial C_X^{(-1)}(t, x | t_0, x_0)}{\partial x_j} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial C_X^{(-1)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial D_v(t, x)}{\partial x_j} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial^2 C_X^{(-1)}(t, x | t_0, x_0)}{\partial x_i \partial x_j} \end{aligned}$$

and

$$G_X^{(0,3)}(t, x | t_0, x_0) = \frac{m}{2},$$

and

$$G_X^{(1)}(t, x | t_0, x_0) = - \frac{\partial C_X^{(0)}(t, x | t_0, x_0)}{\partial t} + G_X^{(1,1)}(t, x | t_0, x_0) + G_X^{(1,2)}(t, x | t_0, x_0) + G_X^{(1,3)}(t, x | t_0, x_0),$$

where

$$\begin{aligned} G_X^{(1,1)}(t, x | t_0, x_0) &= - \sum_{i=1}^m \mu_i(t, x) \frac{\partial C_X^{(0)}(t, x | t_0, x_0)}{\partial x_i} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m \frac{\partial v_{ij}(t, x)}{\partial x_i} \frac{\partial C_X^{(0)}(t, x | t_0, x_0)}{\partial x_j} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial C_X^{(0)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial D_v(t, x)}{\partial x_j} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial^2 C_X^{(0)}(t, x | t_0, x_0)}{\partial x_i \partial x_j}, \end{aligned}$$

$$G_X^{(1,2)}(t, x | t_0, x_0) = \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial C_X^{(0)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial C_X^{(0)}(t, x | t_0, x_0)}{\partial x_j},$$

and

$$\begin{aligned} G_X^{(1,3)}(t, x | t_0, x_0) &= \frac{\partial D_v(t, x)}{\partial t} - \sum_{i=1}^m \frac{\partial \mu_i(t, x)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^2 v_{ij}(t, x)}{\partial x_i \partial x_j} + \sum_{i=1}^m \mu_i(t, x) \frac{\partial D_v(t, x)}{\partial x_i} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^m \frac{\partial v_{ij}(t, x)}{\partial x_i} \frac{\partial D_v(t, x)}{\partial x_j} \\ &\quad - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \left[\frac{\partial^2 D_v(t, x)}{\partial x_i \partial x_j} - \frac{\partial D_v(t, x)}{\partial x_i} \frac{\partial D_v(t, x)}{\partial x_j} \right]. \end{aligned}$$

For $k \geq 2$

$$G_X^{(k)}(t, x | t_0, x_0) = -\frac{\partial C_X^{(k-1)}(t, x | t_0, x_0)}{\partial t} + G_X^{(k,1)}(t, x | t_0, x_0) + G_X^{(k,2)}(t, x | t_0, x_0),$$

where

$$\begin{aligned} G_X^{(k,1)}(t, x | t_0, x_0) &= -\sum_{i=1}^m \mu_i(t, x) \frac{\partial C_X^{(k-1)}(t, x | t_0, x_0)}{\partial x_i} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m \frac{\partial v_{ij}(t, x)}{\partial x_i} \frac{\partial C_X^{(k-1)}(t, x | t_0, x_0)}{\partial x_j} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial C_X^{(k-1)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial D_v(t, x)}{\partial x_j} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial^2 C_X^{(k-1)}(t, x | t_0, x_0)}{\partial x_i \partial x_j} \end{aligned}$$

and

$$\begin{aligned} G_X^{(k,2)}(t, x | t_0, x_0) &= \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \frac{\partial C_X^{(0)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial C_X^{(k-1)}(t, x | t_0, x_0)}{\partial x_j} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t, x) \left[\sum_{h=1}^{k-2} \binom{k-1}{h} \frac{\partial C_X^{(h)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial C_X^{(k-1-h)}(t, x | t_0, x_0)}{\partial x_j} \right]. \end{aligned}$$

Unlike the reducible case, the PDEs in Theorem 2 are in general unsolvable analytically. But we can Taylor-expand each coefficient, $C_X^{(k)}(t, x | t_0, x_0)$ around (t_0, x_0) up to j_k -th order in order to achieve the same approximation error of $O_p(\Delta^{K+1})$ for each coefficient. We denote it as $C_X^{(j_k, k)}(t, x | t_0, x_0)$ where j_k is the maximum order of the Taylor approximation. So, $C_X^{(j_k, k)}(t, x | t_0, x_0)$ can be written as:

$$C_X^{(j_k, k)}(t, x | t_0, x_0) = \sum_{0 \leq a_0 + a_1 + \dots + a_m \leq j_k} c_{0a_0 1a_1 \dots ma_m}^{(k)} \Delta^{a_0} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \dots (x_m - x_{0m})^{a_m}, \quad (31)$$

where a_q 's are non-negative integers, $q = 0, 1, \dots, m$ and we use $c_{0a_0 1a_1 \dots ma_m}^{(k)}$ instead of $c_{0a_0 1a_1 \dots ma_m}^{(k)}(t_0, x_0)$ to simplify expressions. If $a_q = 0$ we omit q -th index, for instance, if $a_q = 0$ for all $0 \leq q \leq m$, $c_{0a_0 1a_1 \dots ma_m}^{(k)} =$

$c^{(k)}$, which is the constant term of $C_X^{(j_k, k)}(t, x | t_0, x_0)$. When $a_q = 1$ we write just subscript q rather than q_1 .

Superseding the coefficients $C_X^{(k)}(t, x | t_0, x_0)$'s of (25) with $C_X^{(j_k, k)}(t, x | t_0, x_0)$ gives the following K -th order log-density function $\tilde{l}_X^{(K)}(\Delta, t, x | t_0, x_0)$.

$$\tilde{l}_X^{(K)}(t, x | t_0, x_0) = -\frac{m}{2} \ln(2\pi\Delta) - D_v(t, x) + \frac{C_X^{(j_{-1}, -1)}(t, x | t_0, x_0)}{\Delta} + \sum_{k=0}^K C_X^{(j_k, k)}(t, x | t_0, x_0) \frac{\Delta^k}{k!} \quad (32)$$

and

$$\left| l_X^{(K)}(t, x | t_0, x_0) - \tilde{l}_X^{(K)}(t, x | t_0, x_0) \right| = O_p(\Delta^{K+1}).$$

It is important to not make $D_v(t, x)$, which is known, part of $C_X^{(0)}(t, x | t_0, x_0)$ and to not Taylor-expand it so that $\tilde{l}_X^{(K)}(t, x | t_0, x_0)$ can also solve the backward equation (27). Now, we just need to find the appropriate order j_k for each coefficient $C_X^{(k)}(t, x | t_0, x_0)$ and the coefficients of the Taylor-expansion $C_X^{(j_k, k)}(t, x | t_0, x_0)$, which are to be discussed below. Let us call what we have discussed the irreducible method.

For each k , j_k is determined to make the approximation error of each term, $C_X^{(j_k, k)}(t, x | t_0, x_0) \Delta^k$ same as that of (25), $O_p(\Delta^{K+1})$. Because $x - x_0 = O_p(\Delta^{1/2})$,

$$\left| C_X^{(k)}(t, x | t_0, x_0) \Delta^k - C_X^{(j_k, k)}(t, x | t_0, x_0) \Delta^k \right| = O_p\left((x - x_0)^{j_k - a_0} \Delta^{a_0} \Delta^k\right) = O_p\left(\Delta^{(j_k - a_0)/2 + a_0 + k}\right).$$

Therefore, we have to set $j_k = 2(K + 1 - k) - a_0$, which can be obtained by solving $(j_k - a_0)/2 + a_0 + k = K + 1$. Notice that the maximum order of $(x - x_0)$, $j_k - a_0$ decreases twice as fast as that of Δ , as a_0 increases. Considering the terms without Δ in the expansion, i.e. $a_0 = 0$, $j_k = 2(K - k)$, which is of course the same as the order for time-homogenous diffusion process. However, having Δ term in the Taylor-expansion of the coefficients makes the maximum order of the terms with Δ less than that of the terms without Δ . For each k , the combinations of $(a_0, |a|)$, where $a = (a_1, \dots, a_m)$, to calculate $\tilde{l}_X^{(K)}$ are as follows:

$k = -1$	$(a_0, a) \in \{(0, 2(K + 2)), (1, 2(K + 1)), (2, 2K), \dots, ((K + 1), 2), ((K + 2), 0)\}$
$k = 0$	$(a_0, a) \in \{(0, 2(K + 1)), (1, 2K), (2, 2(K - 1)), \dots, (K, 2), ((K + 1), 0)\}$
$k = 1$	$(a_0, a) \in \{(0, 2K), (1, 2(K - 1)), (2, 2(K - 2)), \dots, ((K - 1), 2), (K, 0)\}$
\vdots	\vdots
$k = K - 1$	$(a_0, a) \in \{(0, 4), (1, 2), (2, 0)\}$
$k = K$	$(a_0, a) \in \{(0, 2), (1, 0)\}$
$k = K + 1$	$(a_0, a) \in \{(0, 0)\}$

(33)

The reason why we need to include $C_X^{(0, K+1)}$, even though we compute $\tilde{l}_X^{(K)}$, is explained below.

In order to find the Taylor-approximation coefficients of $C_X^{(j_k, k)}(t, x | t_0, x_0)$, we first need to put $C_X^{(j_k, k)}(t, x | t_0, x_0)$ and the j_k -th order Taylor-expansions of $\mu_i(t, x)$, $v_{ij}(t, x)$ and $D_v(t, x)$, respectively, in places of $C_X^{(k)}(t, x | t_0, x_0)$,

$\mu_i(t, x)$, $v_{ij}(t, x)$ and $D_v(t, x)$ in $f_X^{(k-1)}(t, x|t_0, x_0)$. Analogously to (32), $\tilde{f}_X^{(k-1)}(t, x|t_0, x_0)$, $k \geq 0$ denote the corresponding $f_X^{(k-1)}(t, x|t_0, x_0)$ after this replacement. Notice that $G_X^{(k)}(t, x | t_0, x_0)$ depends on $\mu_i(t, x)$, $v_{ij}(t, x)$, and all the former coefficients, $C_X^{(h)}(t, x | t_0, x_0)$, $h \leq k-1$.

In the time-homogeneous case, $C_X^{(j_k, k)}(t, x | t_0, x_0)$'s and $f_X^{(k-1)}(t, x|t_0, x_0)$'s are independent of time variable. Equating the coefficients of the same orders in $\tilde{f}_X^{(-2)}(x|x_0) = 0$ to zeros and solving the simultaneous equations of the coefficients enable us to attain $C_X^{(-1, j-1)}(x | x_0)$. These coefficients must be calculated recursively from low to high order because the coefficients of higher order terms depend on those of lower order terms. Likewise, given $C_X^{(j_{k-1}, -1)}(x | x_0)$, the coefficients of $C_X^{(j_0, 0)}(x | x_0)$ need to be computed from low to high order. Note that the differential equation $f_X^{(k-1)}(x|x_0)$ for $C_X^{(k)}(x | x_0)$ is dependent on $C_X^{(h)}(x | x_0)$, $h \leq k-1$. Therefore, each of the subsequent Taylor-expansions, $C_X^{(j_k, k)}(x | x_0)$'s can be acquired in the same way using $C_X^{(j_h, h)}(x | x_0)$, $h \leq k-1$, and the coefficients of $\tilde{l}_X^{(K)}(x | x_0)$ have to be found recursively in increasing order of Δ .

Proposition 2 Suppose that the diffusion X is reducible and let $l_X^{(K)}$ be its approximate log-transition density function obtained by using the reducible method. If we calculate its log-likelihood expansion, $\tilde{l}_X^{(K)}$ by using irreducible method. Then, each coefficient $C_X^{(j_k, k)}(t, x | t_0, x_0)$ from $\tilde{l}_X^{(K)}$ is a Taylor-expansion in (t, x) about (t_0, x_0) at order j_k of the coefficient $C_X^{(k)}(t, x | t_0, x_0) = C_X^{(k)}[t, \gamma(t, x) | t_0, \gamma(t_0, x_0)]$ from $l_X^{(K)}$.

It is critical to obtain all coefficients of $C_X^{(j_k, k)}(t, x | t_0, x_0)$ at each step of solving simultaneous equations. However, this recursive way of finding the approximate coefficients breaks down if it is time-inhomogeneous, to be precise if σ is a function of t as well as x , since infinitely many coefficients of $C_X^{(j_{k-1}, -1)}(t, x | t_0, x_0)$ and $C_X^{(j_0, 0)}(t, x | t_0, x_0)$ cannot be determined from $\tilde{f}_X^{(-2)}(t, x | t_0, x_0) = 0$ and $\tilde{f}_X^{(-1)}(t, x | t_0, x_0) = 0$, respectively. First, it is clear from $f_X^{(-1)}(t, x|t_0, x_0)$ that $c_{0_{a_0}}^{(0)}$, $a_0 \geq 1$ are unobtainable because $f_X^{(-1)}(t, x|t_0, x_0)$ involves only $\frac{\partial C_X^{(0)}(t, x|t_0, x_0)}{\partial x_j}$, $j = 1, \dots, m$. Therefore we cannot find any of the coefficients of $(t - t_0)^{a_0}$, $c_{0_{a_0}}^{(0)}$, $a_0 \geq 1$, in $C_X^{(j_0, 0)}(t, x | t_0, x_0)$ from $\tilde{f}_X^{(-1)}(t, x|t_0, x_0) = 0$. Next, as is the case for time-homogeneous diffusion, the constant and first order terms of $C_X^{(j_{k-1}, -1)}(t, x | t_0, x_0)$ are zero. But, equating the second order terms of

$\tilde{f}_X^{(-2)}(t, x|t_0, x_0)$ to zeros,

$$\begin{aligned}
& -2 \begin{pmatrix} c_{0_2}^{(-1)} & \frac{1}{2}c_{01}^{(-1)} & \frac{1}{2}c_{02}^{(-1)} & \cdots & \frac{1}{2}c_{0m}^{(-1)} \\ \frac{1}{2}c_{01}^{(-1)} & c_{1_2}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & \cdots & \frac{1}{2}c_{1m}^{(-1)} \\ \frac{1}{2}c_{02}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & c_{2_2}^{(-1)} & \cdots & \frac{1}{2}c_{2m}^{(-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}c_{0m}^{(-1)} & \frac{1}{2}c_{1m}^{(-1)} & \frac{1}{2}c_{2m}^{(-1)} & \cdots & c_{m_2}^{(-1)} \end{pmatrix} \\
& = 4 \begin{pmatrix} \frac{1}{2}c_{01}^{(-1)} & \frac{1}{2}c_{02}^{(-1)} & \cdots & \frac{1}{2}c_{0m}^{(-1)} \\ c_{1_2}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & \cdots & \frac{1}{2}c_{1m}^{(-1)} \\ \frac{1}{2}c_{12}^{(-1)} & c_{2_2}^{(-1)} & \cdots & \frac{1}{2}c_{2m}^{(-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}c_{1m}^{(-1)} & \frac{1}{2}c_{2m}^{(-1)} & \cdots & c_{m_2}^{(-1)} \end{pmatrix} v(t_0, x_0) \begin{pmatrix} \frac{1}{2}c_{01}^{(-1)} & \frac{1}{2}c_{02}^{(-1)} & \cdots & \frac{1}{2}c_{0m}^{(-1)} \\ c_{1_2}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & \cdots & \frac{1}{2}c_{1m}^{(-1)} \\ \frac{1}{2}c_{12}^{(-1)} & c_{2_2}^{(-1)} & \cdots & \frac{1}{2}c_{2m}^{(-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}c_{1m}^{(-1)} & \frac{1}{2}c_{2m}^{(-1)} & \cdots & c_{m_2}^{(-1)} \end{pmatrix}^T.
\end{aligned}$$

Transposition of a matrix a will be written as a^T . From this

$$\begin{pmatrix} c_{1_2}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & \cdots & \frac{1}{2}c_{1m}^{(-1)} \\ \frac{1}{2}c_{12}^{(-1)} & c_{2_2}^{(-1)} & \cdots & \frac{1}{2}c_{2m}^{(-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}c_{1m}^{(-1)} & \frac{1}{2}c_{2m}^{(-1)} & \cdots & c_{m_2}^{(-1)} \end{pmatrix} = -\frac{1}{2}v(t_0, x_0)^{-1} \quad (34)$$

and

$$c_{0_2}^{(-1)} = -\frac{1}{2} \begin{pmatrix} c_{01}^{(-1)} \\ c_{02}^{(-1)} \\ \vdots \\ c_{0m}^{(-1)} \end{pmatrix}^T v(t_0, x_0) \begin{pmatrix} c_{01}^{(-1)} \\ c_{02}^{(-1)} \\ \vdots \\ c_{0m}^{(-1)} \end{pmatrix}.$$

But $c_{01}^{(-1)}, \dots, c_{0m}^{(-1)}$ are indeterminate and so is $c_{0_2}^{(-1)}$. Actually, the coefficients of $(t - t_0)^{a_0}(x_h - x_{0h})$, $c_{0_{a_0 h}}^{(-1)}$, $h = 1, \dots, m$ and $a_0 \geq 1$, cannot be determined from $\tilde{f}_X^{(-2)}(t, x|t_0, x_0) = 0$. This can be shown from looking at the coefficients $c_{0_{a_0 h}}^{(-1)}$ of $C_X^{(j-1, -1)}(t, x|t_0, x_0)$ in $\tilde{f}_X^{(-2)}(t, x|t_0, x_0) = 0$ and using Lemma 1.

Lemma 1

$$\sum_{i=1}^m v_{ij}(t_0, x_0) \frac{\partial \hat{C}_X^{(2, -1)}(t, x|t_0, x_0)}{\partial x_i} = -(x_j - x_{0j}),$$

where $\hat{C}_X^{(2, -1)}(t, x|t_0, x_0)$ is second order terms of the Taylor-expansion of $C_X^{(-1)}(t, x|t_0, x_0)$ around (t_0, x_0) except the terms $(t - t_0)(x_i - x_{0i})$, $i = 1, \dots, m$.

Consider $\tilde{f}_X^{(-2)}(t, x|t_0, x_0) = 0$ to calculate the coefficients of $C_X^{(j-1, -1)}(t, x|t_0, x_0)$ such that

$$2C_X^{(j-1, -1)}(t, x|t_0, x_0) = -\sum_{i=1}^m \sum_{j=1}^m v_{ij}^{(j-1, -2)}(t, x) \frac{\partial C_X^{(j-1, -1)}(t, x|t_0, x_0)}{\partial x_i} \frac{\partial C_X^{(j-1, -1)}(t, x|t_0, x_0)}{\partial x_j}, \quad (35)$$

where $v_{ij}^{(j-1-2)}(t, x)$ is Taylor-approximation of $v_{ij}(t, x)$ about (t_0, x_0) up to $(j-1-2)$ -th order because the constant and first order terms of $C_X^{(j-1, -1)}(t, x | t_0, x_0)$ are zeros. Let us compute $c_{0_{a_0}h}^{(-1)}$ of $C_X^{(j-1, -1)}(t, x | t_0, x_0)$, $a_0 \geq 1$ and $h = 1, \dots, m$. Because of the fact that

$$-\sum_{i=1}^m v_{ih}(t_0, x_0) \frac{\partial \widehat{C}_X^{(2, -1)}(t, x | t_0, x_0)}{\partial x_i} \frac{\partial c_{0_{a_0}h}^{(-1)}(t-t_0)^{a_0}(x_h-x_{0h})}{\partial x_h} = c_{0_{a_0}h}^{(-1)}(t-t_0)^{a_0}(x_h-x_{0h})$$

due to Lemma 1 and the symmetry of the right-hand side of equation (35), $c_{0_{a_0}h}^{(-1)}$ is cancelled out in (35). Thus, none of $c_{0_{a_0}h}^{(-1)}$, $a_0 \geq 1$ and $h = 1, \dots, m$ in $C_X^{(j-1, -1)}(t, x | t_0, x_0)$ can be determined from $\widetilde{f}_X^{(-2)}(t, x | t_0, x_0) = 0$. Although all the coefficients of $(x_1-x_{01})^{a_1}(x_2-x_{02})^{a_2} \dots (x_m-x_{0m})^{a_m}$, $c_{1_{a_1}2_{a_2} \dots m_{a_m}}^{(-1)}$, can be found from $\widetilde{f}_X^{(-2)}(t, x | t_0, x_0) = 0$, generally none of other terms, $c_{0_{a_0}1_{a_1}2_{a_2} \dots m_{a_m}}^{(-1)}$, $a_0 \geq 1$ and $c_{0_{a_0}1_{a_1}2_{a_2} \dots m_{a_m}}^{(k)}$, $k \geq 0$, can be obtained from the differential equations in Theorem 2 because these coefficients depend on those indeterminate coefficients.

This indeterminacy problem does not arise when the volatility term is independent of t because those indeterminate terms turn out to be zero. For a reducible diffusion, if the dispersion matrix σ is a function of only x , the transformation γ would not be dependent on t either. Recall that the log-transition functions for irreducible diffusions have been postulated by using the results from reducible method. Let us go further and postulate the forms of $C_X^{(-1)}(t, x | t_0, x_0)$ and $C_X^{(0)}(t, x | x_0)$ as

$$C_X^{(-1)}(t, x | t_0, x_0) = C_X^{(-1)}(x | x_0) = -\frac{1}{2} \sum_{i=1}^m [\gamma_i(x) - \gamma_i(x_0)]^2 \quad (36)$$

and

$$C_X^{(0)}(t, x | x_0) = \sum_{i=1}^m [\gamma_i(x) - \gamma_i(x_0)] \int_0^1 \mu_{Y_i} \{t, \gamma(x_0) + u[\gamma(x) - \gamma(x_0)]\} du \quad (37)$$

by replacing y and y_0 with $\gamma(x)$ and $\gamma(x_0)$ as Proposition 2 states, respectively, in equations (17) and (18) even though there does not exist such γ for an irreducible diffusion process. Directly Taylor-expanding (36) and (37) about (t_0, x_0) , $c_{0_{a_0}1_{a_1} \dots m_{a_m}}^{(-1)} = 0$ and $c_{0_{a_0}}^{(0)} = 0$ for all $a_0 \geq 1$ and $|a| \geq 0$. Consequently, we can set $c_{0_{a_0}1_{a_1} \dots m_{a_m}}^{(-1)}$ and $c_{0_{a_0}}^{(0)}$ to zeros for all $a_0 \geq 1$ and $|a| \geq 0$ to compute $\widetilde{l}_X^{(K)}(t, x | t_0, x_0; \theta)$.

Instead, we can Taylor-expand $C_X^{(k)}(t, x | t_0, x_0)$ only in x up to j_k -order and write it as $\overline{C}_X^{(j_k, k)}(t, x | x_0)$,

$$\overline{C}_X^{(j_k, k)}(t, x | x_0) = \sum_{0 \leq |a| \leq j_k} c_{1_{a_1} \dots m_{a_m}}^{(k)}(t, x_0) (x_1-x_{01})^{a_1} (x_2-x_{02})^{a_2} \dots (x_m-x_{0m})^{a_m}, \quad (38)$$

where $c_{1_{a_1} \dots m_{a_m}}^{(k)}(t, x_0)$ depends on t and x_0 . Defining $\widetilde{l}_X^{(K)}(t, x | t_0, x_0)$ as

$$\widetilde{l}_X^{(K)}(t, x | t_0, x_0) = -\frac{m}{2} \ln(2\pi\Delta) - D_v(t, x) + \frac{\overline{C}_X^{(j-1, -1)}(t, x | x_0)}{\Delta} + \sum_{k=0}^K \overline{C}_X^{(j_k, k)}(t, x | x_0) \frac{\Delta^k}{k!}, \quad (39)$$

$\widetilde{l}_X^{(K)}$ can be found by using $\widetilde{f}_X^{(k-1)}(t, x | x_0)$ and $\overline{G}_X^{(k)}(t, x | x_0)$ which are obtained after $\overline{C}_X^{(j_k, k)}$ and Taylor-expansions of $\mu_i(t, x)$, $v_{ij}(t, x)$ and $D_v(t, x)$ in x only take the places of $C_X^{(k)}$, $\mu_i(t, x)$, $v_{ij}(t, x)$ and $D_v(t, x)$

in $f_X^{(k-1)}(t, x | t_0, x_0)$ and $G_X^{(k)}(t, x | t_0, x_0)$ in Theorem 2. In addition, $c_{a_1 \dots a_m}^{(-1)}(t, x_0)$ does not depend on t , so it is $c_{1_{a_1} \dots m_{a_m}}^{(-1)}(x_0)$. Furthermore, $c_{1_{a_1} \dots m_{a_m}}^{(-1)}(x_0) = 0$ when $|a| = 0$ and 1, and $c_{1_{a_1} \dots m_{a_m}}^{(0)}(t, x_0)$ can be set to 0 for $|a| = 0$. Therefore, in order to find $\tilde{l}_X^{(K)}$, we can follow exactly the same procedure as the time-homogeneous irreducible case except that now the coefficients of Taylor-expansion are functions of both x_0 and t . Thus, to make the approximation error equal to $O(\Delta^{K+1})$, we can set $j_k = 2(K+1-k)$. In truth, the latter approach produces more accurate log-likelihood expansion than the former since $C_X^{(k)}(t, x | t_0, x_0)$ is not Taylor-expanded in t .

Nevertheless, if the elements of volatility matrix are functions of both t and x then $c_{0_{a_0} h}^{(-1)} \neq 0$ and $c_{0_{a_0}}^{(0)} \neq 0$ in general where $a_0 \geq 1$ and $h = 1, \dots, m$ and we are unable to calculate any of the coefficients of Taylor-expansion $C_X^{(j_k, k)}(t, x | t_0, x_0)$, $k \geq -1$ but $c_{1_{a_1} 2_{a_2} \dots m_{a_m}}^{(-1)}$ of $C_X^{(j_{-1}, -1)}(t, x | t_0, x_0)$. This is because the coefficients of Taylor-expansions $C_X^{(j_k, k)}(t, x | t_0, x_0)$ cannot be determined without knowing the coefficients of lower order terms and $f_X^{(k-1)}(t, x | t_0, x_0)$ pivots on the previous $C_X^{(j_h, h)}(x | x_0)$, $h \leq k-1$. In spite of this, if we look at each coefficient of $\Delta^{a_0} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \dots (x_m - x_{0m})^{a_m}$ in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ rather than that in $C_X^{(j_k, k)}(t, x | t_0, x_0)$, the indeterminacy problem is resolved as Theorem 3 says.

Theorem 3 $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ is free from indeterminate terms, $c_{0_{a_0} h}^{(-1)}$ and $c_{0_{a_0}}^{(0)}$, where $a_0 \geq 1$ and $h = 1, \dots, m$.

The $c_{0_{a_0}}^{(0)} \Delta^{a_0}$ terms of $C_X^{(j_0, 0)}(t, x | t_0, x_0)$ affect a_0 consecutive subsequent coefficients $C_X^{(j_k, k)}(t, x | t_0, x_0)$, $1 \leq k \leq a_0$ through only $\frac{\partial C_X^{(k-1)}(t, x | t_0, x_0)}{\partial t}$ in $G_X^{(k)}(t, x | t_0, x_0)$. Lemma A1 in Appendix proves that $c_{0_{a_0}}^{(0)}$ are cancelled out by these affected parts of $C_X^{(j_k, k)}(t, x | t_0, x_0)$ in $\tilde{l}_X^{(\infty)}$. Using this result and double mathematical induction on a_0 and $|a|$, where $a = (a_1, \dots, a_m)$, the indeterminate parts of the coefficients of $C_X^{(j_k, k)}(t, x | t_0, x_0)$ can be shown to be offset in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ despite the fact that all of $C_X^{(j_k, k)}(t, x | t_0, x_0)$'s of $\tilde{l}_X^{(\infty)}$ are incomputable on account of the indeterminate terms $c_{0_{a_0} h}^{(-1)}$. Precisely speaking, the coefficient of $\Delta^{a_0} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \dots (x_m - x_{0m})^{a_m}$ in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ is $\sum_{k=-1}^{a_0} c_{0_{a_0-k} 1_{a_1} \dots m_{a_m}}^{(k)} \frac{1}{k!}$, which is free of the indeterminate terms although each of $c_{0_{a_0-k} 1_{a_1} \dots m_{a_m}}^{(k)}$ is not. For this reason, the indeterminacy of the coefficients, $c_{0_{a_0} 1_{a_1} \dots m_{a_m}}^{(-1)}$, $a_0 \neq 0$, of $\Delta^{a_0} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \dots (x_m - x_{0m})^{a_m}$ in $C_X^{(j_{-1}, -1)}(t, x | t_0, x_0)$ and that of all subsequent coefficients due to their dependence on $c_{0_{a_0} 1_{a_1} \dots m_{a_m}}^{(-1)}$ is not a problem in computing $\tilde{l}_X^{(K)}(t, x | t_0, x_0)$ as long as we include first $a_0 + 2$ coefficients of $\tilde{l}_X^{(\infty)}$, that is, $C_X^{(j_k, k)}(t, x | t_0, x_0)$, $-1 \leq k \leq a_0$ in $\tilde{l}_X^{(\infty)}$. The pairs of orders $(a_0, |a|)$ for each $C_X^{(j_k, k)}$ to have approximation error equal to $O_p(\Delta^{K+1})$ are tabulated in (33). In reality, we can compute not $\tilde{l}_X^{(\infty)}$ but $\tilde{l}_X^{(K)}$. Even so, $\tilde{l}_X^{(K)}$ does not suffer from the indeterminacy problem once we find $C_X^{(j_k, k)}$ according to $(a_0, |a|)$ in (33) because the indeterminable parts of included $C_X^{(j_k, k)}$ are eliminated in $\tilde{l}_X^{(K)}$. We need to derive $\tilde{l}_X^{(K)}$, and yet the constant term of $C_X^{(j_{K+1}, K+1)}$ must be added so that the indeterminate terms can be removed from $\tilde{l}_X^{(K)}$. Note that when $|a| = 0$, a_0 of $c_{0_{a_0} 1_{a_1} \dots m_{a_m}}^{(-1)}$ must be $K+2$ as can be seen from (33) so we need to include first $K+3$ coefficients of $\tilde{l}_X^{(\infty)}$.

Let us see how to obtain $\tilde{l}_X^{(K)}(t, x | t_0, x_0)$ in practice. The constant and first order terms of $C_X^{(j_{-1}, -1)}(t, x | t_0, x_0)$

are zero. Remember that Theorem 3 says that we can set $c_{0_{a_0}h}^{(-1)} = 0$ and $c_{0_{a_0}}^{(0)} = 0$ for all $a_0 \geq 1$ and $h = 1, \dots, m$ since these are cancelled out in $\tilde{l}_X^{(K)}$ although they are not zeros. So, to find $C_X^{(j-1,-1)}(t, x | t_0, x_0)$, first we can put zeros in $c_{0_h}^{(-1)}$ and $c_{0_2}^{(-1)}$, where $h = 1, \dots, m$ for the reason that $c_{0_{a_0+1}}^{(-1)}$ depends on only $c_{0_{a_0}h}^{(-1)}$. Then, we get the coefficients of the second order terms as (34).

Once $c_{0_{a_0}1_{a_1}\dots m_{a_m}}^{(-1)} 0 \leq a_0 + a_1 + \dots a_m \leq l$ for a combination $(a_0, |a|)$ in (33) are acquired and put in $C_X^{(j-1,-1)}(t, x | t_0, x_0)$, $c_{0_{a_0}1_{a_1}\dots m_{a_m}}^{(-1)}$ for the next pair $(a_0, |a|)$ in (33) with $a_0 + a_1 + \dots a_m = l + 1$, can be found by solving a system of linear equations obtained from equating all corresponding coefficients of the $l + 1$ -th order terms of $\tilde{f}_X^{(-2)}(t, x | t_0, x_0)$ to zero. In doing so, make sure to set $c_{0_h}^{(-1)} = 0$. Next, since the constant term of $C_X^{(j_0,0)}(t, x | t_0, x_0)$ is zero, given $C_X^{(j-1,-1)}(t, x | t_0, x_0)$, we set the first order terms of $\tilde{f}_X^{(-1)}(t, x | t_0, x_0)$ to zero to find $c_h^{(0)}, h = 1, \dots, m$, by solving the resultant system of linear equations. Remember that $c_0^{(0)} = 0$. Similarly, with $C_X^{(j-1,-1)}(t, x | t_0, x_0)$ and $c_{0_{a_0}1_{a_1}\dots m_{a_m}}^{(0)}, 0 \leq a_0 + a_1 + \dots a_m \leq l$ for a couple $(a_0, |a|)$ in (33), $c_{0_{a_0}1_{a_1}\dots m_{a_m}}^{(0)}, a_0 + a_1 + \dots a_m = l + 1$ for the next combination $(a_0, |a|)$ in (33) can be calculated. The same principle applies to all subsequent coefficients. That is to say, once we find $C_X^{(j-1,-1)}(t, x | t_0, x_0), \dots, C_X^{(j_{k-1},k-1)}(t, x | t_0, x_0)$, and $c_{0_{a_0}1_{a_1}\dots m_{a_m}}^{(k)}, 0 \leq a_0 + a_1 + \dots a_m \leq l$ with $(a_0, |a|)$ in (33), we can compute $c_{0_{a_0}1_{a_1}\dots m_{a_m}}^{(k)}, a_0 + a_1 + \dots a_m = l + 1$ for the next couple $(a_0, |a|)$ in (33) successively up to desired order.

Because the indeterminate coefficients, $c_{0_{a_0+1}}^{(-1)}$ and $c_{0_{a_0}h}^{(-1)}$, where $a_0 \geq 1$ and $h = 1, \dots, m$, correspond to the coefficients of Taylor expansions of $c_{a_1\dots a_m}^{(-1)}(t, x_0)$ in $(t - t_0)$ for $|a| = 0$ and 1, respectively, and $c_{0_{a_0}}^{(0)}$ to those of $c_{a_1\dots a_m}^{(0)}(t, x_0)$ in $(t - t_0)$ for $|a| = 0$ where $a_0 \geq 1$, and they are cancelled out in $\tilde{l}_X^{(\infty)}$, we can let $c_{a_1\dots a_m}^{(-1)}(t, x_0) = 0$ for $|a| = 0$ and 1 and $c_{a_1\dots a_m}^{(0)}(t, x_0) = 0$ for $|a| = 0$ when computing $\tilde{l}_X^{(\infty)}$. However, we can find only $\tilde{l}_X^{(K)}$ in practice so we cannot follow this alternative approach.

4 Log-likelihood Function and its Convergence

Even though model (2) is continuous time model, we can observe X_t only at discrete times such that $\{X_t | t = i\Delta \text{ and } i = 0, \dots, n\}$, where Δ is the fixed time interval between two consecutive observations. Due to the first order Markov property of model (2), the log-likelihood function of data can be written as

$$l_n(\theta) = \sum_{i=1}^n l_X(i\Delta, X_{i\Delta} | (i-1)\Delta, X_{(i-1)\Delta}; \theta),$$

where the log-density of the initial observation X_0 is ignored because it is asymptotically negligible. Replacing l_X with the series expansion of log-transition density function $l_X^{(K)}(\tilde{l}_X^{(K)})$ gives approximate log-likelihood function $l_n^{(K)}(\theta)(\tilde{l}_n^{(K)}(\theta))$. It can be shown that $l_X^{(K)}$ and $\tilde{l}_X^{(K)}$ converge to l_X and therefore $l_n^{(K)}(\theta)$ and $\tilde{l}_n^{(K)}(\theta)$ converge to $l_n(\theta)$.

Assume that $\mu_i(t, x; \theta)$ and $\sigma_{ij}(t, x; \theta)$, $i, j = 1, \dots, m$, and their derivatives at all orders are thrice differentiable in θ . Let θ_0 be the true values of the parameter vector θ . For fixed n and Δ , suppose that

$l_n(\theta)$ has a unique MLE $\hat{\theta}_{n,\Delta} \in \Theta$ for θ . The notation $\hat{\theta}_{n,\Delta}^{(K)}$ denotes the approximate MLE of θ computed by maximizing $l_n^{(K)}(\theta)$ ($\tilde{l}_n^{(K)}(\theta)$).

Theorem 4 For all n ,

$$\sup_{\theta \in \Theta} \left| \tilde{l}_n^{(K)}(\theta) - l_n(\theta) \right| \rightarrow 0 \quad (40)$$

in P_{θ_0} -probability as $\Delta \rightarrow 0$. In the reducible case, the same holds for $l_n^{(K)}(\theta)$. The approximate MLE $\hat{\theta}_{n,\Delta}^{(K)}$ exists almost surely and satisfies $\hat{\theta}_{n,\Delta}^{(K)} - \hat{\theta}_{n,\Delta} \rightarrow 0$ in P_{θ_0} -probability as $\Delta \rightarrow 0$.

Furthermore, suppose that as $n \rightarrow \infty$, we have $\hat{\theta}_{n,\Delta}^{(K)} - \theta_0 \rightarrow 0$ in P_{θ_0} -probability and that there exists a sequence of nonsingular $p \times p$ matrices $S_{n,\Delta}$ such that

$$S_{n,\Delta}^{-1} \left(\hat{\theta}_{n,\Delta} - \theta_0 \right) = O_p(1).$$

Then, there exists a sequence $\Delta_n \rightarrow 0$ such that

$$S_{n,\Delta_n}^{-1} \left(\hat{\theta}_{n,\Delta_n}^{(K)} - \hat{\theta}_{n,\Delta_n} \right) = o_p(1). \quad (41)$$

Because $l_X^{(K)}(\tilde{l}_X^{(K)})$ is a Taylor-expansion of l_X about $\Delta = 0$ (and about $x = x_0$), the approximation error in (40) is inconsiderable in a small neighborhood of x_0 . Even if l_X is not analytic outside of the neighborhood, it doesn't cause a big problem because the approximation error is at most polynomial while the probability of X_t being in this region in time Δ is exponentially small.

5 Accuracy of Reducible and Irreducible Methods

Both reducible and irreducible methods are applied to bivariate time-inhomogeneous Ornstein-Uhlenbeck model for benchmarking purposes. Because the true transition function of this process is known we can examine performance of our approximate transition density functions comparing to the true density function in terms of how accurate the ML estimates are. These results are used as a yardstick when the irreducible method is applied to time-inhomogeneous Heston and BDFS models and ML estimates are obtained. These two models are irreducible and the true densities are not available for either of them.

5.1 Bivariate Time-Inhomogeneous Ornstein-Uhlenbeck Model

Consider the d -dimensional diffusion process following

$$dX_t = [A(t) X_t + a(t)] + B(t) dW_t, \quad (42)$$

where $A(t)$ and $B(t)$ are $d \times d$ matrices and $a(t)$ is a $d \times 1$ vector with each element possibly being a function of t . The transition probability of X_t is known to have a d -dimensional normal distribution (see page 149 in Arnold (1974)),

$$N[m_t(t_0, x_0), V_t(t_0, x_0)],$$

where the mean is

$$m_t(t_0, x_0) = C(t, t_0) \left[x_0 + \int_{t_0}^t C(u, t_0)^{-1} a(u) du \right]$$

and the variance is

$$V_t(t_0, x_0) = C(t, t_0) \left[\int_{t_0}^t C(u, t_0)^{-1} B(u) B(u)^T \left[C(u, t_0)^{-1} \right]^T du \right] C(t, t_0)^T.$$

Here $C(t, t_0)$ is the solution of the homogeneous matrix equation

$$\frac{d}{dt} C(t, t_0) = A(t) C(t, t_0), \quad C(t_0, t_0) = I_d.$$

Let $\alpha = [\alpha_i]_{i=1,2}$, $\beta = [\beta_i]_{i=1,2}$, and $k = [k_{i,j}]_{i,j=1,2}$ which is assumed to be full rank. It can be shown that the bivariate time-inhomogeneous Ornstein-Uhlenbeck process,

$$dY_t = k(\alpha + \beta t - Y_t) dt + dW_t \quad (43)$$

has the bivariate normal distribution with mean

$$m_t(t_0, y_0) = \exp(-k\Delta) \left(y_0 - \alpha - \beta t_0 + \int_0^\Delta \exp(ku) k\beta u du \right) + \alpha + \beta t_0$$

and variance-covariance

$$V_t(t_0, y_0) = \int_0^\Delta \exp[-k(\Delta - u)] (\exp[-k(\Delta - u)])^T du.$$

The parameters in (43) cannot be identified on the basis of discrete data. Phillips (1973), Hansen and Sargent (1983) and Kessler and Rahbek (2004) discuss this problem. We will assume that $k_{21} = 0$ then k becomes a triangular matrix with real elements and it can have real eigenvalues. Then the mapping $k \rightarrow \exp(-k\Delta)$ is invertible and the parameters can be identified from discrete data. The process Y_t is a unit diffusion so we can use the reducible method to find the coefficients of the likelihood expansion.

To my best knowledge (42) is the only multivariate model for which true transition density function is available. We transform Y_t into X_t using $(X_{1t}, X_{2t})^T = (\exp(Y_{1t}), \exp(Y_{2t}))^T$ to apply the irreducible method directly to the transformed process. Applying Ito's lemma, the resulting process is

$$dX_t = \begin{pmatrix} X_{1t} \left[\frac{1}{2} + k_{11}(\alpha_1 + \beta_1 t - \ln(X_{1t})) + k_{12}(\alpha_2 + \beta_2 t - \ln(X_{2t})) \right] \\ X_{2t} \left[\frac{1}{2} + k_{21}(\alpha_1 + \beta_1 t - \ln(X_{1t})) + k_{22}(\alpha_2 + \beta_2 t - \ln(X_{2t})) \right] \end{pmatrix} dt + \begin{pmatrix} X_{1t} & 0 \\ 0 & X_{2t} \end{pmatrix} dW_t. \quad (44)$$

True log-transition function of X_t can be obtained by change of variable because the transition density of Y_t is known i.e.

$$l_X(t, x|t_0, x_0) = -\ln(x_1 x_2) + l_Y[t, \ln(x_1), \ln(x_2)|t_0, \ln(x_{01}), \ln(x_{02})].$$

Therefore model (44) can serve as a benchmark for further investigation of the performance of our approximation methods in the next two subsections where irreducible method is applied to two irreducible time-inhomogeneous diffusions.

Table 1: Monte Carlo simulation for bivariate time-inhomogeneous Ornstein-Uhlenbeck model

Parameter	θ_0	$\hat{\theta}^{(True)} - \theta_0$		$\hat{\theta}^{(True)} - \hat{\theta}^{(2, Redu)}$		$\hat{\theta}^{(True)} - \hat{\theta}^{(2, Irre)}$	
		Bias	Std. Dev	Mean	Std. Dev	Mean	Std. Dev
α_1	0	-0.0033	0.13	0.0000005	0.00006	0.00006	0.0001
α_2	0	0.0016	0.063	-0.000002	0.00004	0.00003	0.00005
β_1	0.001	0.0006	0.023	-0.0000002	0.00001	-0.0000003	0.00002
β_2	0.005	-0.0002	0.011	-0.00000004	0.000008	0.00000001	0.000009
k_{11}	5	0.72	1.15	0.013	0.0089	0.011	0.010
k_{12}	1	0.053	1.67	0.010	0.018	0.0012	0.0038
k_{22}	10	0.65	1.54	0.074	0.033	0.018	0.019

Using the true transition density function of the process (44), 1000 samples of 500 weekly observations ($\Delta = 1/52$) were generated with the initial values $x_{01} = 0.0002$ and $x_{02} = 0.0004$ and the true parameter values in the 2nd column. Denote the ML estimates of parameter when using the true density as $\hat{\theta}^{(True)}$, the approximate density by the reducible method with $K = 2$ as $\hat{\theta}^{(2, Redu)}$, and the approximate density by the irreducible method with $K = 2$ as $\hat{\theta}^{(2, Irre)}$. The 3rd and 4th columns respectively show average of $\hat{\theta}^{(True)} - \theta_0$ and standard deviation of $\hat{\theta}^{(True)}$. Mean and standard deviations of $\hat{\theta}^{(True)} - \hat{\theta}^{(2, Redu)}$ and $\hat{\theta}^{(True)} - \hat{\theta}^{(2, Irre)}$ are shown in the 5th and 6th columns and the 7th and 8th columns, respectively.

Table 1 summarizes Monte Carlo simulation study results for model (44). The true transition density function of the process (44) has been used to generate 1000 samples of 500 weekly observations ($\Delta = 1/52$) with the initial values $x_{01} = 0.0002$ and $x_{02} = 0.0004$ and true parameter values $(\alpha_1, \alpha_2, \beta_1, \beta_2, k_{11}, k_{12}, k_{22}) = (0, 0, 0.001, 0.005, 5, 1, 10)$. Using the true density function and two approximate density functions obtained respectively by the reducible and irreducible methods, we have estimated parameters by MLE. Denote parameter estimates when using the true density as $\hat{\theta}^{(True)}$, the approximate density by the reducible method with $K = 2$ as $\hat{\theta}^{(2, Redu)}$, and the approximate density by the irreducible method with $K = 2$ as $\hat{\theta}^{(2, Irre)}$. Sample means and standard deviations of $\hat{\theta}^{(2, Redu)} - \hat{\theta}^{(True)}$ and $\hat{\theta}^{(2, Irre)} - \hat{\theta}^{(True)}$ are much smaller than those of $\hat{\theta}^{(True)} - \theta_0$ caused by the sampling noise, where θ_0 is the true parameter. For the sake of comparison, we have chosen the same data frequency ($\Delta = 1/52$), sample size, and approximation order $K = 2$ as Aït-Sahalia (2008). Our findings and his results in Table 2 are very much alike. The results of simulation studies suggest that both the reducible and irreducible methods provide a good approximation to the true density function of model (44) and they can be employed when the true density is unknown as is often the case.

5.2 Extended Heston's Stochastic Volatility Model

Heston (1993) suggested the following stochastic volatility model

$$d \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} rX_{1t} \\ a(b - X_{2t}) \end{pmatrix} dt + \begin{pmatrix} \sqrt{(1-\rho^2)X_{2t}}X_{1t} & \rho\sqrt{X_{2t}}X_{1t} \\ 0 & \sigma\sqrt{X_{2t}} \end{pmatrix} dW_t.$$

We have extended this model by making the volatility process X_{2t} follow the Extended Cox-Ingersoll-Ross (ECIR) model of Maghsoodi (1996). Expressing the resulting stochastic differential equation in terms of return of X_{1t} , $x_{1t} = \ln(X_{1t})$ and X_{2t} we get the following Extended Heston (EHeston) model:

$$d \begin{pmatrix} x_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} r - \frac{1}{2}X_{2t} \\ a(b_t - X_{2t}) \end{pmatrix} dt + \begin{pmatrix} \sqrt{(1-\rho^2)X_{2t}} & \rho\sqrt{X_{2t}} \\ 0 & \sigma_1 e^{\sigma_2 t} \sqrt{X_{2t}} \end{pmatrix} dW_t, \quad (45)$$

where $b_t = (\sigma_1^2 d / 4a) \exp(2\sigma_2 t)$ and d is a positive integer.

The irreducible method has been used to find an approximate log-transition density function since model (45) is irreducible unless $\sigma_1 = 0$ and X_{2t} is constant. Because the true joint transition density function of (x_{1t}, X_{2t}) is unknown in general, 1,000 samples of 500 weekly ($\Delta = 7/365$), 500 daily ($\Delta = 1/252$), and 5,000 daily pairs of observations were generated by Euler approximation. To generate each set of simulated data, the process was initialized by $(x_{10}, X_{20}) = (\ln(100), 0.05625)$ with the sampling interval equal to $\Delta/30$ and $a = 3$, $\sigma_1 = 0.5$, $\sigma_2 = 0.001$, $\rho = -0.8$, $r = 0.03$, and $d = 9$. After casting away first 500 pairs of (x_{1t}, X_{2t}) , every 30th pair of simulated data was set aside to build data sets of frequency Δ . In this and next Monte Carlo experiments, we followed the same strategies of generating data and chose the same approximation order $K = 1$ as Aït-Sahalia and Kimmel (2007). We have carried out MLE for each of these simulated data sets after holding r and d fixed.

Bias and standard deviation of $\hat{\theta}^{(1, Irre)}$ for three different data sets are tabulated in Table 2. Overall, the biases and standard errors are relatively small if we compare these with the true values. The 500 weekly results are comparable with third and fourth columns of Table 1 where the true transition density function and 500 weekly simulated observations are used. Comparing 500 weekly and 500 daily cases, the a and σ_2 parameters are estimated less accurately in the latter case. This can be because of the bias discussed in Li et al. (2004), which might have more to do with daily data with sample sizes of 500. If we increase the number of observations to 5,000 for the daily case, standard errors of all $\hat{\theta}^{(1, Irre)}$ decrease significantly as expected. Aït-Sahalia and Kimmel (2007) report analogous results for the Heston model in Table 1.

5.3 Extended BDFS Model

The BDFS model suggested by Balduzzi et al. (1996) extends the univariate diffusion model of the short rate to a three-factor model by introducing a stochastic long-run mean and a volatility. In order to see the

Table 2: Monte Carlo simulation for EHeston model

Parameters	True value	500 Weekly obs.		500 Daily obs.		5,000 Daily obs.	
	θ_0	Bias	Std. dev.	Bias	Std. dev.	Bias	Std. dev.
a	3.00	-0.040	0.29	0.068	0.57	0.016	0.17
σ_1	0.50	-0.0082	0.026	0.0012	0.023	-0.0008	0.0074
σ_2	0.001	0.0013	0.0085	-0.0008	0.040	0.0019	0.0013
ρ	-0.80	-0.0032	0.015	-0.0022	0.013	-0.0008	0.0043

1,000 samples of 500 weekly ($\Delta = 7/365$), 500 daily ($\Delta = 1/252$), and 5,000 daily pairs of observations were generated by Euler approximation for the EHeston model. To generate each set of simulated data, the process was initialized by $(x_{10}, X_{20}) = (\ln(100), 0.05625)$ with the true values of parameters in the 2nd column. For each set of data, ML estimates, $\hat{\theta}^{(1, Irre)}$ are obtained from maximizing the approximate density obtained by the irreducible method with $K=1$. The Bias column displays average of $\hat{\theta}^{(1, Irre)} - \theta_0$ and the Std dev column shows the standard deviation of $\hat{\theta}^{(1, Irre)}$.

performance of our approximation, the BDFS model is generalized by using the Hull-White model for the second factor as follows

$$dX_t = \begin{pmatrix} k_1 (X_{2t} - X_{1t}) \\ k_2 (\alpha_2 + \beta_2 t - X_{2t}) \\ k_3 (\alpha_3 - X_{3t}) \end{pmatrix} dt + \begin{pmatrix} \sqrt{(1 - \rho^2) X_{3t}} & 0 & \rho \sqrt{X_{3t}} \\ 0 & \sigma_{21} e^{\sigma_{22} t} & 0 \\ 0 & 0 & \sigma_{31} \sqrt{X_{3t}} \end{pmatrix} dW_t.$$

This process is irreducible and the true transition density function is unavailable in general so the irreducible method can be applied to obtain an approximate transition density function. Again, we adopted the same data frequency $\Delta = 7/365$, $1/252$, number of observations $n = 500$, 5000 , and approximation order $K = 1$ as the EHeston case. Initial values are set at $X_{10} = 0.05$, $X_{20} = 0.06$, and $X_{30} = 0.03$ to produce data at the frequency of $\Delta/30$ with $k_1 = 10$, $k_2 = 7$, $k_3 = 3$, $\alpha_2 = 0.06$, $\beta_2 = 0.001$, $\alpha_3 = 0.1$, $\sigma_{21} = 0.03$, $\sigma_{22} = 0.001$, $\sigma_{31} = 0.05$, and $\rho = 0.5$. After removing first 500 observations, every 30th triple of data was stored for later use in conducting MLE.

Table 3 shows biases and standard errors of MLE estimator $\hat{\theta}^{(1, Irre)}$. Overall, both the biases and standard errors of the parameter estimators are small after taking true values into account. Like the results of EHeston model, for some parameters of the 500 daily case have been estimated less accurately than those of the 5000 daily case. However, in general, both the biases and standard errors of the parameters have declined substantially for the 5000 daily case.

We have found that biases and standard errors of ML estimators calculated from using our approximate transition density are comparable to those obtained from using true transition density. Hence, we can

Table 3: Monte Carlo simulation for EBDFS model

Parameters	True value	500 Weekly obs.		500 Daily obs.		5,000 Daily obs.	
		Bias	Std. dev.	Bias	Std. dev.	Bias	Std. dev.
k_1	10	1.73	2.29	4.25	7.18	1.15	1.35
k_2	7	0.68	1.37	3.68	3.77	0.28	0.89
k_3	3	0.10	0.70	-0.092	0.62	-0.10	0.34
α_2	0.06	0.00026	0.0027	0.00005	0.0060	0.00005	0.0019
β_2	0.001	0.000009	0.00050	-0.0001	0.0053	0.0000001	0.00016
α_3	0.1	0.0034	0.0021	0.0041	0.0056	0.0028	0.0012
σ_{21}	0.03	0.0001	0.0019	0.00004	0.0019	0.00004	0.00060
σ_{22}	0.001	-0.00006	0.011	0.0018	0.054	0.00007	0.0017
σ_{31}	0.05	0.0001	0.0016	-0.0001	0.0015	-0.0000005	0.00049
ρ	0.5	0.0099	0.033	0.0050	0.031	0.0008	0.010

1,000 samples of 500 weekly ($\Delta = 7/365$), 500 daily ($\Delta = 1/252$), and 5,000 daily triples of observations were generated by Euler approximation for the EBDFS model. To generate each set of simulated data, the process was initialized by $(X_{10}, X_{20}, X_{30}) = (0.050, 0.06, 0.03)$ with the true values of parameters in the 2nd column. For each set of data, ML estimates, $\hat{\theta}^{(1, Irre)}$ are obtained from maximizing the approximate density obtained by the irreducible method with $K=1$. The Bias column displays average of $\hat{\theta}^{(1, Irre)} - \theta_0$ and the Std dev column shows the standard deviation of $\hat{\theta}^{(1, Irre)}$.

conclude that both the irreducible and reducible methods can yield an accurate approximation to true but unknown transition density function of a time-inhomogeneous diffusion even with $K = 1$.

6 Conclusion

Diffusion models have been extensively used in modeling the dynamics of many economic variables. Researchers propose multivariate time-inhomogeneous diffusion models to capture time dependence of the underlying data generating process. This article developed a way to find a very accurate approximate transition density function of a multivariate time-inhomogeneous diffusion in closed-form. In order to check the performance of our methods, we have applied both the reducible and irreducible methods to the bivariate time-inhomogeneous Ornstein-Uhlenbeck model which serves as a benchmark since its true transition density function is known. For further investigation of our methods, the irreducible method has been applied to EHeston and EBDfS models. The outcomes of these Monte Carlo exercises show that both the irreducible and reducible methods can yield an accurate approximation to the true transition density function of a time-inhomogeneous diffusion even with a low order expansion and it can be used when the true transition density function is unavailable as is often the case. The closed-form approximate log-transition density function can be used not only to conduct MLE but also to get a posterior distribution for the parameters of a diffusion process when applying Bayesian methods and to generate simulated data to adopt indirect inference or simulation methods, or efficient method of moment (EMM).

Of course, our methods can be applied to a real data set. Because a time-inhomogeneous diffusion is more general and encompasses time-homogeneous processes, we can let the data say whether or not there is time dependence in the data. By doing so, if we can find an evidence for time-inhomogeneity in the data, taking the time-homogeneity into account might be able to improve, for example, on explaining dynamics of the data and on assets pricing. But a time-inhomogeneous diffusion is not a stationary process. Therefore, unlike the stationary case, the appropriate convergence rate of the ML estimator for a time-inhomogeneous diffusion process is in general different from \sqrt{n} and it can be dependent on the true parameter value and has to be found on a model by model basis. Moreover, the asymptotic distribution of the ML estimator can often be non-normal and include nuisance parameters (Basawa and Scott (1983) and Jeganathan (1995)). Empirical application for the time-inhomogeneous diffusion is left for future work.

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A Proofs

Proof of Proposition 1

Proof. Because the reducibility condition has to do with only the state variables vector x not t , Proposition 1 can be proved in the same way as Proposition 1 in Aït-Sahalia (2008). ■

Proof of Theorem 1

Proof. Let $F_Y^{(K)}(t, y|t_0, y_0)$ and $B_Y^{(K)}(t, y|t_0, y_0)$ be, respectively, the difference between the left- and right-hand sides of (22) and (23) when we substitute (16) for $l_Y(t, y | t_0, y_0)$. It is well known that the log-transition density function of a diffusion process is the solution of the Kolmogorov forward and backward partial differential equations. Therefore, it is sufficient to verify that $F_Y^{(K)}(t, y|t_0, y_0)$ and $B_Y^{(K)}(t, y|t_0, y_0)$ are of order Δ^K in order to prove Theorem 1. Collecting terms of $F_Y^{(K)}(t, y|t_0, y_0)$ and $B_Y^{(K)}(t, y|t_0, y_0)$ in increasing orders of Δ , we obtain

$$F_Y^{(K)}(t, y|t_0, y_0) = \sum_{k=-2}^{K-1} f_Y^{(k)}(t, y|t_0, y_0) \frac{\Delta^k}{k!} + O(\Delta^K) \quad (46)$$

and

$$B_Y^{(K)}(t, y|t_0, y_0) = \sum_{k=-2}^{K-1} b_Y^{(k)}(t, y|t_0, y_0) \frac{\Delta^k}{k!} + O(\Delta^K) \quad (47)$$

with the convention that $(-2)! = (-1)! = 0! = 1$. We will show that $f_Y^{(k)}(t, y|t_0, y_0) = 0$ and $b_Y^{(k)}(t, y|t_0, y_0) = 0$ for all $k = -2, -1, \dots, K-1$ are satisfied by the coefficients given in Theorem 1. In fact, those coefficients can be found by solving the partial differential equations of the coefficients, $f_Y^{(k)}(t, y|t_0, y_0) = 0$ and $b_Y^{(k)}(t, y|t_0, y_0) = 0$.

First,

$$C_Y^{(-1)}(t, y | t_0, y_0) = -\frac{1}{2} \sum_{i=1}^m \left[\frac{\partial C_Y^{(-1)}(t, y | t_0, y_0)}{\partial y_i} \right]^2 \quad (48)$$

from $f_Y^{(-2)}(t, y|t_0, y_0) = 0$. Since the density goes to a Gaussian as $\Delta \rightarrow 0$, the solution $C_Y^{(-1)}(t, y | t_0, y_0)$ to (48) must have a strict maximum at $y = y_0$ such that

$$C_Y^{(-1)}(t, y | t_0, y_0) = -\frac{1}{2} \sum_{i=1}^m (y_i - y_{0i})^2. \quad (49)$$

Given (49),

$$f_Y^{(-1)}(t, y|t_0, y_0) = -\sum_{i=1}^m (y_i - y_{0i}) \mu_{Y_i}(t, y) + \sum_{i=1}^m (y_i - y_{0i}) \frac{\partial C_Y^{(0)}(t, y | t_0, y_0)}{\partial y_i}.$$

Solving $f_Y^{(-1)}(t, y|t_0, y_0) = 0$ for $C_Y^{(0)}(t, y | y_0, s)$,

$$C_Y^{(0)}(t, y | t_0, y_0) = \sum_{i=1}^m (y_i - y_{0i}) \int_0^1 \mu_{Y_i}[t, y_0 + u(y - y_0)] du + \sum_{i,j=1, j \neq i}^m \alpha_{ij}^{(0)} \frac{y_i - y_{0i}}{y_j - y_{0j}} + M_Y^{(0)},$$

where $\alpha_{ij}^{(0)}$ and $M_Y^{(0)}$ are independent of y because they are integration constants. Moreover $\alpha_{ij}^{(0)} = 0$ for all $i, j = 1, \dots, m$ ($j \neq i$) due to the boundary condition that $C_Y^{(0)}(t, y | t_0, y_0)$ is finite when $y_j = y_{0j}$ for all $j = 1, \dots, m$. Since $C_Y^{(0)}(t, y | t_0, y_0)$ should also satisfy

$$b_Y^{(-1)}(t, y | t_0, y_0) = - \sum_{i=1}^m \mu_{Y_i}(t, y) (y_i - y_{0i}) - \sum_{i=1}^m (y_i - y_{0i}) \frac{\partial C_Y^{(0)}(t, y | t_0, y_0)}{\partial y_{0i}} = 0$$

it must be

$$\sum_{i=1}^m (y_i - y_{0i}) \frac{\partial M_Y^{(0)}}{\partial y_{0i}} = 0$$

for all y and y_0 and therefore $M_Y^{(0)}$ is constant. Because $p_Y \rightarrow \phi(y)$ as $\Delta \rightarrow 0$ and

$$\lim_{\Delta \rightarrow 0} \left[l_Y^{(K)}(t, y | t_0, y_0) + \frac{m}{2} \ln(2\pi\Delta) + \frac{1}{2\Delta} \sum_{i=1}^m (y_i - y_{0i})^2 \right] = C_Y^{(0)}(t, y | t_0, y_0),$$

$M_Y^{(0)} = 0$ so that the limiting density integrates to one rather than $\exp(M_Y^{(0)})$ as Δ goes to zero.

Next, for $k = 0$,

$$f_Y^{(0)}(t, y | t_0, y_0) = C_Y^{(1)}(t, y | t_0, y_0) + \sum_{i=1}^m (y_i - y_{0i}) \frac{\partial C_Y^{(1)}(t, y | t_0, y_0)}{\partial y_i} - G_Y^{(1)}(t, y | t_0, y_0),$$

where $G_Y^{(1)}(t, y | t_0, y_0)$ depends on $C_Y^{(0)}(t, y | t_0, y_0)$ and $\mu_Y(t, y)$ as can be seen in (20). The partial differential equation $f_Y^{(0)}(t, y | t_0, y_0) = 0$ of $C_Y^{(1)}(t, y | t_0, y_0)$ can be similarly solved as follows:

$$C_Y^{(1)}(t, y | t_0, y_0) = \int_0^1 G_Y^{(1)}(t, y_0 + u(y - y_0) | t_0, y_0) du + \sum_{i,j=1, j \neq i}^m \alpha_{ij}^{(1)} \frac{y_i - y_{0i}}{(y_j - y_{0j})^2} + M_Y^{(1)}.$$

Again the integration constants $\alpha_{ij}^{(1)}$ are zero because $C_Y^{(1)}(t, y | t_0, y_0)$ is finite when passing through the axes $y_j = y_{0j}$ for all $j = 1, \dots, m$. Using the fact that $C_Y^{(1)}(t, y | t_0, y_0)$ must satisfy $b_Y^{(0)}(t, y | t_0, y_0) = 0$ too, we have

$$M_Y^{(1)} - \sum_{i=1}^m (y_i - y_{0i}) \frac{\partial M_Y^{(1)}}{\partial y_{0i}} = 0. \quad (50)$$

The unique solution $M_Y^{(1)}$ for (50) for all y and y_0 is $M_Y^{(1)} = 0$, which gives the coefficient $C_Y^{(1)}(t, y | t_0, y_0)$.

In general, for all $k \geq 1$

$$f_Y^{(k-1)}(t, y | t_0, y_0) = C_Y^{(k)}(t, y | t_0, y_0) + \frac{1}{k} \sum_{i=1}^m (y_i - y_{0i}) \frac{\partial C_Y^{(k)}(t, y | t_0, y_0)}{\partial y_i} - G_Y^{(k)}(t, y | t_0, y_0)$$

where $G_Y^{(k)}(t, y | t_0, y_0)$ is given in (21) and depends on the previously determined coefficients $C_Y^{(0)}(t, y | t_0, y_0), \dots, C_Y^{(k-1)}(t, y | t_0, y_0)$ and $\mu_Y(t, y)$. If we solve the differential equation, $f_Y^{(k-1)}(t, y | t_0, y_0) = 0$ for $C_Y^{(k)}(t, y | t_0, y_0)$ we obtain

$$C_Y^{(k)}(t, y | t_0, y_0) = k \int_0^1 G_Y^{(k)}(t, y_0 + u(y - y_0) | t_0, y_0) u^{k-1} du + \sum_{i,j=1, j \neq i}^m \alpha_{ij}^{(k)} \frac{y_i - y_{0i}}{(y_j - y_{0j})^{k+1}} + M_Y^{(k)}.$$

The same boundary condition as for $C_Y^{(0)}(t, y | t_0, y_0)$ and $C_Y^{(1)}(t, y | t_0, y_0)$ and the fact that $b_Y^{(k-1)}(t, y | t_0, y_0) = 0$ make the integration constants $\alpha_{ij}^{(k)} = 0$ for all $i, j = 1, \dots, m$ ($i \neq j$) and $M_Y^{(k)} = 0$, respectively, for all $k \geq 1$, which yields the solution (19).

Hence, with those coefficients $C_Y^{(k)}(t, y | t_0, y_0)$'s in the theorem, $f_Y^{(k-1)}(t, y | t_0, y_0) = 0$ and $b_Y^{(k-1)}(t, y | t_0, y_0) = 0$ hold for all k , $-1 \leq k \leq K$ and so $F_Y^{(K)}(t, y | t_0, y_0) = O(\Delta^K)$ and $B_Y^{(K)}(t, y | t_0, y_0) = O(\Delta^K)$. This proves that (16) solves the Kolmogorov equations to the order Δ^K . Theorem 1 verifies that (16) is actually the right expression for the $K - 1$ -th order Taylor expansion of log-density of the process Y_t in Δ . ■

Proof of Theorem 2

Proof. As in the proof of Theorem 1, let $F_X^{(K)}(t, x | t_0, x_0)$ and $B_X^{(K)}(t, x | t_0, x_0)$ be the difference between the left- and right-hand sides of (26) and (27), respectively, when $l_X(t, x | t_0, x_0)$ is replaced by (25). Rearranging terms of $F_X^{(K)}(t, x | t_0, x_0)$ and $B_X^{(K)}(t, x | t_0, x_0)$ in increasing orders of Δ , we get

$$F_X^{(K)}(t, x | t_0, x_0) = \sum_{k=-2}^{K-1} f_X^{(k)}(t, x | t_0, x_0) \frac{\Delta^k}{k!} + O(\Delta^K)$$

and

$$B_X^{(K)}(t, x | t_0, x_0) = \sum_{k=-2}^{K-1} b_X^{(k)}(t, x | t_0, x_0) \frac{\Delta^k}{k!} + O(\Delta^K)$$

with the convention that $(-2)! = (-1)! = 0! = 1$. $f_Y^{(k)}(t, x | t_0, x_0)$'s are given in Theorem 2 and set to zeros to get $F_X^{(K)}(t, x | t_0, x_0) = O(\Delta^K)$. It is important to not make $D_v(t, x)$, which is known, part of $C_X^{(0)}(t, x | t_0, x_0)$ so that $B_X^{(K)}(t, x | t_0, x_0) = O(\Delta^K)$ can be held. ■

Proof of Proposition 2

Proof. Suppose the diffusion X is reducible then there exists $\gamma(t, x)$ and $C_X^{(k)}(t, x | t_0, x_0) = C_Y^{(k)}[t, \gamma(t, x) | t_0, \gamma(t_0, x_0)]$. By construction, the coefficients of $C_X^{(j_k, k)}(t, x | t_0, x_0)$ acquired from the irreducible method are the coefficients of direct Taylor-expansion of $C_X^{(k)}[t, \gamma(t, x) | t_0, \gamma(t_0, x_0)]$ in (t, x) about (t_0, x_0) at order j_k for each k . ■

Proof of Lemma 1

Proof. Using (34) and the symmetry of $v(t_0, x_0)$

$$\begin{aligned} & \sum_{i=1}^m v_{ij}(t_0, x_0) \frac{\partial \widehat{C}_X^{(2, -1)}(t, x | t_0, x_0)}{\partial x_i} \\ &= \sum_{i=1}^m v_{ij}(t_0, x_0) \left[\sum_{h=1}^{i-1} c_{hi}^{(-1)}(t_0, x_0) (x_h - x_{0h}) + 2c_{i2}^{(-1)}(t_0, x_0) (x_i - x_{0i}) + \sum_{h=i+1}^m c_{ih}^{(-1)}(t_0, x_0) (x_h - x_{0h}) \right] \\ &= \sum_{i=1}^m v_{ji}(t_0, x_0) \left[- \sum_{h=1}^m v_{ih}^{-1}(t_0, x_0) (x_h - x_{0h}) \right] \\ &= - (x_j - x_{0j}) \end{aligned}$$

for each j . ■

Lemma A1 $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ is free of the indeterminate terms, $c_{0_{a_0}}^{(0)}$, $a_0 \geq 1$.

Proof. Let us Taylor-expand $C_X^{(k)}(t, x | t_0, x_0)$ with respect to x only up to j_k -th order. Then

$$\overline{C}_X^{(j_k, k)}(t, x | x_0) = c^{(k)}(t, x_0) + \sum_{1 \leq |a| \leq j_k} c_{1_{a_1} \dots m_{a_m}}^{(k)}(t, x_0) (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m},$$

where $a = (a_1, a_2, \dots, a_m)$. It is obvious that Taylor-expanding $\overline{C}_X^{(j_k, k)}(t, x | x_0)$ around t_0 yields $C_X^{(j_k, k)}(t, x | t_0, x_0)$ that includes all indeterminate terms, $c_{0_{a_0}}^{(0)} \Delta^{a_0}$, $a_0 \geq 1$ in $C_X^{(j_0, 0)}(t, x | t_0, x_0)$. The constant term of $\overline{C}_X^{(j_0, 0)}(t, x | x_0)$, $c^{(0)}(t, x_0)$, affects all subsequent terms of $c^{(k)}(t, x_0)$ in $\overline{C}_X^{(j_k, k)}(t, x | x_0)$, $k \geq 1$ through only $\frac{\partial \overline{C}_X^{(k-1)}(t, x | x_0)}{\partial t}$ in $\overline{C}_X^{(k)}$. Therefore $c^{(k)}(t, x_0)$, $k \geq 1$ is equal to $(-1)^k \frac{\partial^k c^{(0)}(t, x_0)}{\partial t^k}$ plus other terms involving t only but not x . Taylor-expanding $c^{(0)}(t, x_0)$ in t around t_0 ,

$$c^{(0)}(t, x_0) = \sum_{i=1}^{\infty} \frac{\partial^i c^{(0)}(t_0, x_0)}{\partial t^i} \frac{\Delta^i}{i!}$$

because the constant term of $C_X^{(j_0, 0)}(t, x | t_0, x_0)$ is zero. Here we abuse notation by using $\frac{\partial^i c^{(0)}(t_0, x_0)}{\partial t^i}$ instead of $\left. \frac{\partial^i c^{(0)}(t, x_0)}{\partial t^i} \right|_{t=t_0}$. Note that $\frac{\partial^i c^{(0)}(t_0, x_0)}{\partial t^i} = c_{0_i}^{(0)}$. Therefore the part of $\overline{C}_X^{(j_k, k)}(t, x | x_0)$ influenced by the term $c^{(0)}(t, x_0)$ is

$$\begin{aligned} (-1)^k \frac{\partial^k c^{(0)}(t, x_0)}{\partial t^k} &= (-1)^k \sum_{i=1}^{\infty} \frac{\partial^i c^{(0)}(t_0, x_0)}{\partial t^i} \frac{\partial^k}{\partial t^k} \left[\frac{\Delta^i}{i!} \right] \\ &= (-1)^k \sum_{i \geq k} \frac{\partial^i c^{(0)}(t_0, x_0)}{\partial t^i} \frac{\Delta^{i-k}}{(i-k)!} \end{aligned}$$

for $k \geq 1$.

Because

$$\tilde{l}_X^{(K)}(t, x | t_0, x_0) = -\frac{m}{2} \ln(2\pi\Delta) - D_v(x, t; \theta) + \frac{\overline{C}_X^{(j-1, -1)}(t, x | x_0)}{\Delta} + \sum_{k=0}^K \overline{C}_X^{(j_k, k)}(t, x | x_0) \frac{\Delta^k}{k!},$$

collecting all of such terms depending on $c^{(0)}(t, x_0)$ in $\overline{C}_X^{(j_k, k)}(t, x | x_0)$, $k \geq 0$,

$$\begin{aligned} \sum_{k=0}^{\infty} c^{(k)}(t, x | x_0; \theta) \frac{\Delta^k}{k!} &= \sum_{k=0}^{\infty} (-1)^k \sum_{i \geq k} \frac{\partial^i c^{(0)}(t_0, x_0)}{\partial t^i} \frac{\Delta^{i-k}}{(i-k)!} \frac{\Delta^k}{k!} \\ &= \sum_{i=0}^{\infty} \frac{\partial^i c^{(0)}(t_0, x_0)}{\partial t^i} \frac{\Delta^i}{i!} \sum_{k=0}^i \frac{i!}{(i-k)!k!} (-1)^k \\ &= 0 \end{aligned}$$

in $\tilde{l}_X^{(\infty)}$. The last equality is because $\sum_{k=0}^i \frac{i!}{(i-k)!k!} (-1)^k = (-1+1)^i = 0$ for all $i \geq 1$ and $c^{(0)}(t_0, x_0) = 0$. This implies that $c_{0_{a_0}}^{(0)}$, $a_0 \geq 1$, are cancelled out by a_0 consecutive subsequent coefficients $C_X^{(j_k, k)}(t, x | t_0, x_0)$,

$1 \leq k \leq a_0$. In consequent, as long as we compute $C_X^{(j_k, k)}$ according to (33) $\tilde{l}_X^{(K)}(t, x | t_0, x_0)$ is free of the indeterminate terms of $C_X^{(j_0, 0)}$. ■

Lemma A2 The coefficients of $(x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$, $a = (a_1, \dots, a_m)$ and $|a| \geq 1$ in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ are not dependent upon the indeterminate terms of $C_X^{(j_{-1}, -1)}(t, x | t_0, x_0)$.

Proof. The $(x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ terms in $\tilde{l}_X^{(\infty)}$ are acquired from the first two coefficients, $\frac{c_{01a_1 2a_2 \dots ma_m}^{(-1)} \Delta (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m}}{\Delta} + c_{1a_1 2a_2 \dots ma_m}^{(0)} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m}$. Hence it is sufficient to verify that $c_{01a_1 2a_2 \dots ma_m}^{(-1)} + c_{1a_1 2a_2 \dots ma_m}^{(0)}$ are the indeterminate terms free for all $|a| \geq 1$. This can be done by mathematical induction on $|a|$ in the similar way to the proof of Lemma A5 using Lemma 1 so every detail is omitted. The differences between the two are the fact that only first two terms matter and none of the L , P , and N terms are involved in this case. ■

Lemma A3 The coefficients of $\Delta (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$, $a = (a_1, \dots, a_m)$ and $|a| \geq 1$ in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ are not dependent upon the indeterminate terms of $C_X^{(j_{-1}, -1)}(t, x | t_0, x_0)$.

Proof. The coefficients of $\Delta (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ of $\tilde{l}_X^{(\infty)}$ are attained from the first three coefficients $\frac{c_{021a_1 2a_2 \dots ma_m}^{(-1)} \Delta^2 (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m}}{\Delta} + c_{01a_1 2a_2 \dots ma_m}^{(0)} \Delta (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m} + c_{1a_1 2a_2 \dots ma_m}^{(1)} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m} \Delta$. Given Lemma 1 and Lemma A2, we can show that $c_{021a_1 2a_2 \dots ma_m}^{(-1)} + c_{01a_1 2a_2 \dots ma_m}^{(0)} + c_{1a_1 2a_2 \dots ma_m}^{(1)}$ are not dependent on any of indeterminate terms of $C_X^{(j_{-1}, -1)}(t, x | t_0, x_0)$ by induction on $|a|$ in the same way as we prove Lemma A5 below except that $N^{(1,2)}$ term is irrelevant here. In order to save space further proof is not included. ■

Lemma A4 $\sum_{k=i}^p \frac{(m+i-1)!}{(m+k)!} (-1)^{k+1-i} \frac{(p-i)!}{(p-k)!} = -\frac{1}{p+m}$ for all integer i , $1 \leq i \leq p$ and a non-negative integer m .

Proof. It can be proved by double mathematical induction on i and m . First, consider the case $i = 1$ and $m = 0$.

$$\begin{aligned} \sum_{k=1}^p \frac{1}{k!} (-1)^k \frac{(p-1)!}{(p-k)!} &= \frac{1}{p} \sum_{k=1}^p (-1)^k \frac{p!}{(p-k)!} \frac{1}{k!} \\ &= \frac{1}{p} \left[\sum_{k=0}^p (-1)^k \frac{p!}{(p-k)!} \frac{1}{k!} - 1 \right] \\ &= -\frac{1}{p} \end{aligned}$$

because $\sum_{k=0}^p (-1)^k \frac{p!}{(p-k)!} \frac{1}{k!} = (-1 + 1)^p = 0$. Suppose that $\sum_{k=h}^p \frac{(h-1)!}{k!} (-1)^{k+1-h} \frac{(p-h)!}{(p-k)!} = -\frac{1}{p}$ for $i =$

$h, 1 \leq h < p$ and $m = 0$. Then, for $i = h + 1$ and $m = 0$,

$$\begin{aligned}
\sum_{k=h+1}^p \frac{h!}{k!} (-1)^{k-h} \frac{(p-h-1)!}{(p-k)!} &= \sum_{k=h}^p \frac{h!}{k!} (-1)^{k-h} \frac{(p-h-1)!}{(p-k)!} - \frac{1}{(p-h)} \\
&= -\frac{h}{(p-h)} \sum_{k=h}^p \frac{(h-1)!}{k!} (-1)^{k+1-h} \frac{(p-h)!}{(p-k)!} - \frac{1}{(p-h)} \\
&= \frac{h}{(p-h)p} - \frac{1}{(p-h)} \\
&= -\frac{1}{p}.
\end{aligned}$$

Now assume that $\sum_{k=h}^p \frac{(l+h-1)!}{(l+k)!} (-1)^{k+1-h} \frac{(p-h)!}{(p-k)!} = -\frac{1}{p+l}$ for $i = h, 1 \leq h < p$ and for $m = l \geq 0$. The last step is to show that it holds for $i = h$ and $m = l + 1$. If we let $k + 1 = j$,

$$\begin{aligned}
\sum_{k=h}^p \frac{(l+h)!}{(l+1+k)!} (-1)^{k+1-h} \frac{(p-h)!}{(p-k)!} &= \sum_{j=h+1}^{p+1} \frac{(l+h)!}{(l+j)!} (-1)^{j-h} \frac{(p-h)!}{(p+1-j)!} \\
&= \frac{-(l+h)}{(p+1-h)} \sum_{j=h+1}^{p+1} \frac{(l+h-1)!}{(l+j)!} (-1)^{j+1-h} \frac{(p+1-h)!}{(p+1-j)!} \\
&= \frac{-(l+h)}{(p+1-h)} \left[\sum_{j=h}^{p+1} \frac{(l+h-1)!}{(l+j)!} (-1)^{j+1-h} \frac{(p+1-h)!}{(p+1-j)!} + \frac{1}{(l+h)} \right] \\
&= \frac{-(l+h)}{(p+1-h)} \left[-\frac{1}{(p+1+l)} + \frac{1}{(l+h)} \right] \\
&= -\frac{1}{(p+l+1)}.
\end{aligned}$$

■

Lemma A5 *The coefficients of $\Delta^{a_0} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$, $a_0 \geq 2$ and $a = (a_1, \dots, a_m)$, $|a| \geq 1$ in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ are not dependent upon the indeterminate terms of $C_X^{(j-1, -1)}(t, x | t_0, x_0)$.*

Proof. Notice that the coefficients of $\Delta^{a_0} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ in $\tilde{l}_X^{(\infty)}$ are determined by the first $a_0 + 2$ coefficients. Thus, it suffices to show that

$$\sum_{k=-1}^{a_0} c_{0_{a_0-k} 1_{a_1} \cdots m_{a_m}}^{(k)} \frac{1}{k!} = R_{0_{a_0+1} 1_{a_1} \cdots m_{a_m}} \quad (51)$$

for all $a_0 \geq 2$ and $a = (a_1, \dots, a_m)$, $|a| \geq 1$. $R_{0_{a_0+1} 1_{a_1} \cdots m_{a_m}}$ represents the coefficient of $\Delta^{a_0} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ in $\tilde{l}_X^{(\infty)}$, which is free from the indeterminate terms. The subindices of R that coincides with those of $c^{(-1)}$ tell us whose coefficient it is. In this paper, R without any indices denotes a term that does not depend on the indeterminate terms. Because the same pattern of $G_X^{(k)}$ repeats from $k = 2$ on, this can be proved by double mathematical induction on a_0 and $|a|$. First, the $a_0 = 2$ and $|a| \geq 1$ case of (51) can be verified by induction on $|a|$ and using Lemma 1, Lemma A2, Lemma A3 and Lemma A4 similarly to the last step of double mathematical induction described below and the explanations are left out here.

Now, assume that (51) holds for $a_0 = p \geq 2$ and $|a| \geq 1$, that is $\sum_{k=-1}^{p-1} c_{0_{p-k-1}1_{a_1}\dots m_{a_m}}^{(k)} \frac{1}{k!} = R_{0_p1_{a_1}\dots m_{a_m}}$. Then we can justify that (51) is also valid for $a_0 = p+1$ and $|a| \geq 1$ by induction on $|a|$. Using Lemma 1, Lemma A2, Lemma A3, Lemma A4 and the assumption $\sum_{k=-1}^{p-1} c_{0_{p-k-1}1_{a_1}\dots m_{a_m}}^{(k)} \frac{1}{k!} = R_{0_p1_{a_1}\dots m_{a_m}}$, the $a_0 = p+1$ and $|a| = 1$ case can be attested by showing that the coefficient of $\Delta^p(x_1 - x_{01})$ in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ does not suffer from the indeterminacy problem without loss of generality. The rest of this inductive step is very close to the last part of this induction so they are excluded.

Suppose that (51) is satisfied for $a_0 = p+1$ and $1 \leq |a| \leq n$, where $a = (a_1 - 1, a_2, \dots, a_m)$. We complete the proof if we can show that (51) also holds for $a_0 = p+1$ and $|a| = n+1$, where $a = (a_1, a_2, \dots, a_m)$ without losing generality. Note that to verify the $a_0 = p+1$ and $|a| = n+1$ case we use induction on $|a|$ because the fact that (51) is true for $a_0 = p+1$ and $1 \leq |a| \leq n$ is required in addition to the assumption and the previous lemmas.

$$\begin{aligned}
& \text{Considering the coefficient of } \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \text{ in } \tilde{f}_X^{(-2)}(t, x | t_0, x_0) = 0, \\
& -2c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
& = 2 \sum_{l=1}^m \sum_{i=1}^m v_{il}(t_0, x_0) \frac{\partial \tilde{C}_X^{(2,-1)}(t, x | t_0, x_0)}{\partial x_i} \frac{c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}}{\partial x_l} \\
& + 2Q_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + R \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}, \text{ where} \\
& Q_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
& = \frac{1}{2} \sum_{e_0=1}^p \sum_{b_0=1}^p \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=1}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0}(x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \quad \left. \frac{\partial c_{0_{b_0}1_{b_1}\dots m_{b_m}}^{(-1)} \Delta^{b_0}(x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0}1_{e_1}\dots m_{e_m}}^{(-1)} \Delta^{e_0}(x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \\
& + \sum_{e_0=1}^{p+1} \sum_{d_0=0}^p \sum_{|e|=0}^{n+1} \sum_{|b|=1}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0}(x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \quad \left. \frac{\partial c_{0_{b_0}1_{b_1}\dots m_{b_m}}^{(-1)} \Delta^{b_0}(x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0}1_{e_1}\dots m_{e_m}}^{(-1)} \Delta^{e_0}(x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \cdot Q_{0_{a_0}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)}
\end{aligned}$$

indicates the rest of indeterminate coefficient of $\Delta^{a_0}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$ in the second term of the right hand side of (28).

In the following proof, $L_{0_{a_0}1_{a_1}\dots m_{a_m}}^{(k)}$, $P_{0_{a_0}1_{a_1}\dots m_{a_m}}^{(k)}$ and $N_{0_{a_0}1_{a_1}2_{a_2}\dots m_{a_m}}^{(k)}$ will be used to denote the coefficients of $\Delta^{a_0}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$ which depends on the indeterminate terms of $C_X^{(j-1,-1)}$ for different parts of $\tilde{f}_X^{(k-1)}$, $k \geq 0$.

Because of Lemma 1, it is reduced to $(|a| - 1) c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} = Q_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + R \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$. Therefore

$$Q_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} = (|a| - 1) c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} + R. \quad (52)$$

Next, find the term $\Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$ in $\tilde{f}_X^{(-1)}(t, x | t_0, x_0) = 0$. Then,

$$\begin{aligned}
& - \sum_{l=1}^m \sum_{i=1}^m v_{il}(t_0, x_0) \frac{\partial \widehat{C}_X^{(2,-1)}(t, x|t_0, x_0)}{\partial x_i} \frac{\partial c_{0p1a_1 \dots m_{a_m}}^{(0)} \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}}{\partial x_1} \\
& - L_{0p1a_1 \dots m_{a_m}}^{(0)} \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} = - \frac{\partial c_{0p+11a_1 \dots m_{a_m}}^{(-1)} \Delta^{p+1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}}{\partial t} \\
& + P_{0p1a_1 \dots m_{a_m}}^{(0)} \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + R \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}.
\end{aligned}$$

From the first term of (29) we get $L_{0p1a_1 \dots m_{a_m}}^{(0)} \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} =$

$$\begin{aligned}
& \sum_{e_0=0}^p \sum_{b_0=0}^p \sum_{d_0=0}^p \sum_{e|e=0}^{n+1} \sum_{b|b=0}^{n+1} \sum_{d|d=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0}(x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \frac{\partial c_{0b_01b_1 \dots mb_m}^{(-1)} \Delta^{b_0}(x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_01e_1 \dots me_m}^{(0)} \Delta^{e_0}(x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right]
\end{aligned}$$

and the $\Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$ term from $\widetilde{G}_X^{(0,1)}$ is $P_{0p1a_1 \dots m_{a_m}}^{(0)} \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$

$$\begin{aligned}
& = - \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{e|e=0}^{n+1} \sum_{b|b=0}^{n+1} \sum_{i=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \mu_i(t_0, x_0) \Delta^{b_0}(x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \left. \frac{\partial c_{0e_01e_1 \dots me_m}^{(-1)} \Delta^{e_0}(x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i})}{\partial x_i} \right] \\
& + \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{e|e=0}^{n+1} \sum_{b|b=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \left[\frac{\partial v_{ij}(t_0, x_0)}{\partial x_i} \right] \Delta^{b_0}(x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \left. \frac{\partial c_{0e_01e_1 \dots me_m}^{(-1)} \Delta^{e_0}(x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \\
& - \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{e|e=0}^{n+1} \sum_{b|b=0}^{n+1} \sum_{d|d=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0}(x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \left. \frac{\partial c_{0e_01e_1 \dots me_m}^{(-1)} \Delta^{e_0}(x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial^{d_0+d_1+\dots+d_m} \left[\frac{\partial D_{ij}(t_0, x_0)}{\partial x_j} \right] \Delta^{d_0}(x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \right] \\
& + \frac{1}{2} \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{e|e=0}^{n+1} \sum_{b|b=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0}(x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \left. \frac{\partial c_{0e_01e_1 \dots me_m}^{(-1)} \Delta^{e_0}(x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i})(x_j - x_{0j})}{\partial x_i \partial x_j} \right].
\end{aligned}$$

Due to Lemma 1, it can be simplified as $|a| c_{0p1i_1 2i_2 \dots mi_m}^{(0)} \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$

$$\begin{aligned}
& - L_{0p1a_1 2a_2 \dots ma_m}^{(0)} \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
& = - (p+1) c_{0p+11a_1 2a_2 \dots ma_m}^{(-1)} \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
& + P_{0p1a_1 2a_2 \dots ma_m}^{(0)} \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + R \Delta^p(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \text{ or}
\end{aligned}$$

$$c_{0p1i_1 2i_2 \dots mi_m}^{(0)} = \frac{1}{|a|} \left[- (p+1) c_{0p+11a_1 2a_2 \dots ma_m}^{(-1)} + L_{0p1a_1 2a_2 \dots ma_m}^{(0)} + P_{0p1a_1 2a_2 \dots ma_m}^{(0)} \right] + R. \quad (53)$$

Looking at the term $\Delta^{p-1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$ in $\widetilde{f}_X^{(0)}(t, x|t_0, x_0) = 0$,

$$\begin{aligned}
& c_{0p-11a_1 2a_2 \dots ma_m}^{(1)} \Delta^{p-1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
& - \sum_{l=1}^m \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t_0, x_0) \frac{\partial C_X^{(-1,2)}(t, x|t_0, x_0)}{\partial x_i} \frac{\partial c_{0p-11a_1 \dots ma_m}^{(1)} \Delta^{p-1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}}{\partial x_i} \\
& - L_{0p-11a_1 2a_2 \dots ma_m}^{(1)} \Delta^{p-1}(x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\partial c_{0p-1a_1 2a_2 \dots m a_m}^{(0)} \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}}{\partial t} + P_{0p-1a_1 2a_2 \dots m a_m}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
&+ N_{0p-1a_1 2a_2 \dots m a_m}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + R \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}, \text{ where} \\
&L_{0p-1a_1 2a_2 \dots m a_m}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} = \\
&\sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{p-1} \sum_{|b|=0}^{p-1} \sum_{|d|=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\left. \frac{\partial c_{0b_0 1e_1 \dots m e_m}^{(-1)} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_0 1e_1 \dots m e_m}^{(1)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right], \text{ which is}
\end{aligned}$$

from the second term of $\tilde{f}_X^{(0)}$.

Similarly to $P_{0p-1a_1 2a_2 \dots m a_m}^{(0)}$, the coefficient of $\Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$ in $\tilde{G}_X^{(1,1)}$ is

$$\begin{aligned}
&P_{0p-1a_1 2a_2 \dots m a_m}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
&= -\sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{|e|=0}^{p-1} \sum_{|b|=0}^{p-1} \sum_{i=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \mu_i(t_0, x_0) \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
&\left. \frac{\partial c_{0e_0 1e_1 \dots m e_m}^{(0)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i})}{\partial x_i} \right] \\
&+ \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{|e|=0}^{p-1} \sum_{|b|=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \left[\frac{\partial v_{ij}(t_0, x_0)}{\partial x_i} \right] \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
&\left. \frac{\partial c_{0e_0 1e_1 \dots m e_m}^{(0)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \\
&- \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{p-1} \sum_{|b|=0}^{p-1} \sum_{|d|=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
&\left. \frac{\partial c_{0e_0 1e_1 \dots m e_m}^{(0)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial^{d_0+d_1+\dots+d_m} \left[\frac{\partial D_v(t_0, x_0)}{\partial x_j} \right] \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \right] \\
&+ \frac{1}{2} \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{|e|=0}^{p-1} \sum_{|b|=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
&\left. \frac{\partial c_{0e_0 1e_1 \dots m e_m}^{(0)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i}) (x_j - x_{0j})}{\partial x_i \partial x_j} \right] \text{ and from } \tilde{G}_X^{(1,2)} \text{ we obtain}
\end{aligned}$$

$$\begin{aligned}
&N_{0p-1a_1 2a_2 \dots m a_m}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
&= \frac{1}{2} \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{p-1} \sum_{|b|=0}^{p-1} \sum_{|d|=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\left. \frac{\partial c_{0b_0 1b_1 \dots m b_m}^{(0)} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_0 1e_1 \dots m e_m}^{(0)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right]. \text{ Notice that}
\end{aligned}$$

$\tilde{G}_X^{(1,3)}$ is independent of the indeterminate terms.

By Lemma 1, it can be written as

$$\begin{aligned}
&(|a| + 1) c_{0p-1a_1 2a_2 \dots m a_m}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} - L_{0p-1a_1 2a_2 \dots m a_m}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
&= -p c_{0p-1a_1 2a_2 \dots m a_m}^{(0)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + P_{0p-1a_1 2a_2 \dots m a_m}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}
\end{aligned}$$

$+N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)}\Delta^{p-1}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m}+R\Delta^{p-1}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m}$. Hence

$$c_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} = \frac{1}{(|a|+1)} \left[-pc_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(0)} + L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} \right] + R. \quad (54)$$

Now, turn to the term $\Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m}$ in $\tilde{f}_X^{(k-1)}(t, x|t_0, x_0) = 0$ for $2 \leq k \leq p$.

$$\begin{aligned} & c_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m} - \\ & \frac{1}{k} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t_0, x_0) \frac{\partial C_X^{(-1,2)}(t, x|t_0, x_0)}{\partial x_i} \frac{\partial c_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m}}{\partial x_1} - \\ & L_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m} \\ & = -\frac{\partial c_{0_{p-k+1}1_{a_1}\dots m_{a_m}}^{(k-1)} \Delta^{p-k+1}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m}}{\partial t} + P_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m} + \\ & N_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m} + R\Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m}. \end{aligned}$$

Similarly to before, we get $L_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)}$ from the second part of $\tilde{f}_X^{(k-1)}$, $P_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)}$ from $\tilde{G}_X^{(k,1)}$, and $N_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)}$ from $\tilde{G}_X^{(k,2)}$. Here

$$\begin{aligned} & L_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m} = \\ & \frac{1}{k} \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0}(x_1-x_{01})^{d_1}\dots(x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ & \quad \left. \begin{array}{l} e_0+b_0+d_0=p-k \\ e+b+d=a \end{array} \right] \\ & \quad \text{If } e_0=p-k \text{ then } |b|>1 \text{ or } |d|>0 \\ & \frac{\partial c_{0_{b_0}1_{b_1}\dots m_{b_m}}^{(-1)} \Delta^{b_0}(x_1-x_{01})^{b_1}\dots(x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0}1_{e_1}\dots m_{e_m}}^{(k)} \Delta^{e_0}(x_1-x_{01})^{e_1}\dots(x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \Big], \end{aligned}$$

$$\begin{aligned} & P_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m} = \\ & \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \mu_i(t_0, x_0) \Delta^{b_0}(x_1-x_{01})^{b_1}\dots(x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\ & \quad \left. \begin{array}{l} e_0+b_0=p-k \\ e+b=a \end{array} \right] \\ & \frac{\partial c_{0_{e_0}1_{e_1}\dots m_{e_m}}^{(k-1)} \Delta^{e_0}(x_1-x_{01})^{e_1}\dots(x_m-x_{0m})^{e_m} (x_i-x_{0i})}{\partial x_i} \\ & + \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \left[\frac{\partial v_{ij}(t_0, x_0)}{\partial x_i} \right] \Delta^{b_0}(x_1-x_{01})^{b_1}\dots(x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\ & \quad \left. \begin{array}{l} e_0+b_0=p-k \\ e+b=a \end{array} \right] \\ & \frac{\partial c_{0_{e_0}1_{e_1}\dots m_{e_m}}^{(k-1)} \Delta^{e_0}(x_1-x_{01})^{e_1}\dots(x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \\ & - \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{n+1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0}(x_1-x_{01})^{b_1}\dots(x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\ & \quad \left. \begin{array}{l} e_0+b_0+d_0=p-k \\ e+b+d=a \end{array} \right] \\ & \frac{\partial c_{0_{e_0}1_{e_1}\dots m_{e_m}}^{(k-1)} \Delta^{e_0}(x_1-x_{01})^{e_1}\dots(x_m-x_{0m})^{e_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial^{d_0+d_1+\dots+d_m} \left[\frac{\partial D_v(t_0, x_0)}{\partial x_j} \right] \Delta^{d_0}(x_1-x_{01})^{d_1}\dots(x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \\ & + \frac{1}{2} \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0}(x_1-x_{01})^{b_1}\dots(x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\ & \quad \left. \begin{array}{l} e_0+b_0=p-k \\ e+b=a \end{array} \right] \\ & \frac{\partial c_{0_{e_0}1_{e_1}\dots m_{e_m}}^{(k-1)} \Delta^{e_0}(x_1-x_{01})^{e_1}\dots(x_m-x_{0m})^{e_m} (x_i-x_{0i})(x_j-x_{0j})}{\partial x_i \partial x_j} \Big] \text{ and} \end{aligned}$$

$$N_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \Delta^{p-k}(x_1-x_{01})^{a_1}\dots(x_m-x_{0m})^{a_m} =$$

$$\begin{aligned}
& \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \frac{\partial c_{0b_0 1b_1 \dots mb_m}^{(0)} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0e_0 1e_1 \dots me_m}^{(k-1)} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \\
& + \frac{1}{2} \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left\{ \frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \frac{\partial c_{0b_0 1b_1 \dots mb_m}^{(h)} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0e_0 1e_1 \dots me_m}^{(k-1-h)} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right\}.
\end{aligned}$$

Then

$$\begin{aligned}
& \left[1 + \frac{1}{k} |a| \right] c_{0p-k 1a_1 \dots ma_m}^{(k)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} - \\
& L_{0p-k 1a_1 \dots ma_m}^{(k)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} = \\
& - (p-k+1) c_{0p-k+1 1a_1 \dots ma_m}^{(k-1)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} + \\
& P_{0p-k 1a_1 \dots ma_m}^{(k)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} + N_{0p-k 1a_1 \dots ma_m}^{(k)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} + \\
& R \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \text{ by using Lemma 1. Thus}
\end{aligned}$$

$$c_{0p-k 1a_1 \dots ma_m}^{(k)} = \frac{k}{k+|a|} \left[L_{0p-k 1a_1 \dots ma_m}^{(k)} - (p-k+1) c_{0p-k+1 1a_1 \dots ma_m}^{(k-1)} + P_{0p-k 1a_1 \dots ma_m}^{(k)} + N_{0p-k 1a_1 \dots ma_m}^{(k)} \right] + R \quad (55)$$

for $2 \leq k \leq p$.

The fact that $\sum_{k=-1}^r c_{0p-k 1a_1 \dots ma_m}^{(k)} \frac{1}{k!} = R_{0r+1 1a_1 \dots ma_m}$, for $0 \leq r \leq p-1$ and $|a| \geq 1$ and the assumption $\sum_{k=-1}^p c_{0p-k 1a_1 \dots ma_m}^{(k)} \frac{1}{k!} = R_{0p+1 1a_1 \dots ma_m}$, where $p \geq 3$ and $1 \leq |a| \leq n$ play important roles in deriving the following results.

$$\begin{aligned}
& P_{0p 1a_1 \dots ma_m}^{(0)} \Delta^p (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \\
& = - \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \mu_i(t_0, x_0) \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \left. \frac{\partial \left[\sum_{k=0}^{e_0-1} \left(-c_{0e_0-(k+1) 1e_1 \dots me_m}^{(k)} \frac{1}{k!} + R_{0e_0-k 1e_1 \dots me_m} \right) \right] \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_i-x_{0i})}{\partial x_i} \right] \\
& + \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \left[\frac{\partial v_{ij}(t_0, x_0)}{\partial x_i} \right] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \left. \frac{\partial \left[\sum_{k=0}^{e_0-1} \left(-c_{0e_0-(k+1) 1e_1 \dots me_m}^{(k)} \frac{1}{k!} + R_{0e_0-k 1e_1 \dots me_m} \right) \right] \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \\
& - \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \left. \frac{\partial \left[\sum_{k=0}^{e_0-1} \left(-c_{0e_0-(k+1) 1e_1 \dots me_m}^{(k)} \frac{1}{k!} + R_{0e_0-k 1e_1 \dots me_m} \right) \right] \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_i-x_{0i})}{\partial x_i} \right] \times
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{\partial^{d_0+d_1+\dots+d_m} \left[\frac{\partial D_v(t_0, x_0)}{\partial x_j} \right] \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \right] \\
& + \frac{1}{2} \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \quad \left. \frac{\left[\sum_{k=0}^{e_0-1} \left(-c_{e_0-(k+1)1e_1 \dots m e_m}^{(k)} i j \frac{1}{k!} + R_{0e_0-k1e_1 \dots m e_m} i j \right) \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i})(x_j - x_{0j})}{\partial x_i \partial x_j} \right] \\
& = \sum_{k=0}^{p-1} \sum_{e_0=1+k}^p \sum_{b_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \mu_i(t_0, x_0) \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \quad \left. \frac{\partial \left[\left(c_{e_0-(k+1)1e_1 \dots m e_m}^{(k)} i \frac{1}{k!} \right) \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i})}{\partial x_i} \right] \\
& - \sum_{k=0}^{p-1} \sum_{e_0=1+k}^p \sum_{b_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} \left[\frac{\partial v_{ij}(t_0, x_0)}{\partial x_i} \right] \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \quad \left. \frac{\partial \left[\left(c_{e_0-(k+1)1e_1 \dots m e_m}^{(k)} j \frac{1}{k!} \right) \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \\
& + \sum_{k=0}^{p-1} \sum_{e_0=1+k}^p \sum_{b_0=0}^{p-k-1} \sum_{d_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^m \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \quad \left. \frac{\partial \left[\left(c_{e_0-(k+1)1e_1 \dots m e_m}^{(k)} i \frac{1}{k!} \right) \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i})}{\partial x_i} \times \right. \\
& \quad \left. \frac{\partial^{d_0+d_1+\dots+d_m} \left[\frac{\partial D_v(t_0, x_0)}{\partial x_j} \right] \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \right] \\
& - \frac{1}{2} \sum_{k=0}^{p-1} \sum_{e_0=1+k}^p \sum_{b_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{b_0+b_1+\dots+b_m} v_{ij}(t_0, x_0) \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \dots \partial x_m^{b_m}} \times \right. \\
& \quad \left. \frac{\left[\left(c_{e_0-(k+1)1e_1 \dots m e_m}^{(k)} i j \frac{1}{k!} \right) \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_i - x_{0i})(x_j - x_{0j})}{\partial x_i \partial x_j} \right] + R \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
& = - \sum_{k=1}^p P_{0_{p-k}1_{a_1} \dots m_{a_m}}^{(k)} \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \frac{1}{(k-1)!} + R \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}. \text{ There-} \\
& \text{fore}
\end{aligned}$$

$$P_{0_{p-1}1_{a_1} \dots m_{a_m}}^{(0)} = - \sum_{k=1}^p P_{0_{p-k}1_{a_1} \dots m_{a_m}}^{(k)} \frac{1}{(k-1)!} + R. \quad (56)$$

$$\begin{aligned}
& \text{For } 2 \leq k \leq p-1, N_{0_{p-k}1_{a_1} \dots m_{a_m}}^{(k)} \Delta^{p-k} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} = \\
& \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \quad \left. \frac{\partial c_{0_{b_0}1_{b_1} \dots m_{b_m}}^{(0)} i \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0}1_{e_1} \dots m_{e_m}}^{(k-1)} j \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \\
& + \frac{1}{2} \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left\{ \frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right.
\end{aligned}$$

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$$\begin{aligned}
& + \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \frac{\partial [R_{b_0+1 \ 1_{b_1} \dots m_{b_m} i}] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} 1_{e_1} \dots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \\
& + \frac{1}{2} \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left\{ \frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \left[\sum_{h=1}^{k-2} \binom{k-1}{h} \frac{\partial c_{0_{b_0} 1_{b_1} \dots m_{b_m} i} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} 1_{e_1} \dots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \right\} \\
& = N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,1)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} + N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,2)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} + \\
& R_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m}, \text{ where } N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,1)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \\
& = - \sum_{e_0=0}^{p-k} \sum_{b_0=1}^{p-k+1} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \frac{\partial [c_{0_{b_0} 1_{b_1} \dots m_{b_m} i}^{(-1)}] \Delta^{b_0-1} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} 1_{e_1} \dots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \\
& - \frac{1}{2} \sum_{e_0=0}^{p-k-1} \sum_{b_0=1}^{p-k} \sum_{d_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \frac{\partial [\sum_{l=1}^{b_0} c_{0_{b_0-l} 1_{b_1} \dots m_{b_m} i}^{(l)}] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} 1_{e_1} \dots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right], \\
& N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,2)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \\
& = - \frac{1}{2} \sum_{e_0=0}^{p-k-1} \sum_{b_0=1}^{p-k} \sum_{d_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \frac{\partial [\sum_{l=1}^{b_0} c_{0_{b_0-l} 1_{b_1} \dots m_{b_m} i}^{(l)}] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} 1_{e_1} \dots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \\
& + \frac{1}{2} \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left\{ \frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \left[\sum_{h=1}^{k-2} \binom{k-1}{h} \frac{\partial c_{0_{b_0} 1_{b_1} \dots m_{b_m} i} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} 1_{e_1} \dots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \right\}
\end{aligned}$$

and

$$\begin{aligned}
& R_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \\
& = \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \frac{\partial [R_{b_0+1 \ 1_{b_1} \dots m_{b_m} i}] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} 1_{e_1} \dots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right].
\end{aligned}$$

So,

$$N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k)} = N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,1)} + N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,2)} + R_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k)} \quad (57)$$

for $2 \leq k \leq p-1$.

When $k = p$,

$$N_{1_{a_1} \dots m_{a_m}}^{(p)} = N_{1_{a_1} \dots m_{a_m}}^{(p,1)} + N_{1_{a_1} \dots m_{a_m}}^{(p,2)} + R_{1_{a_1} \dots m_{a_m}}^{N_{1_{a_1} \dots m_{a_m}}^{(p)}} \quad (58)$$

but $N_{1_{a_1} \dots m_{a_m}}^{(p,1)} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$

$$= - \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_1+\dots+d_m} v_{ij}(t_0, x_0) (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ \left. \frac{\partial \left[c_{01b_1 \dots mb_{m,i}}^{(-1)} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \right]}{\partial x_i} \frac{\partial c_{1e_1 \dots me_{m,j}}^{(p-1)} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right],$$

$$N_{1_{a_1} \dots m_{a_m}}^{(p,2)} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$$

$$= \frac{1}{2} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left\{ \frac{\partial^{d_1+\dots+d_m} v_{ij}(t_0, x_0) (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ \left. \left[\sum_{h=1}^{p-2} \binom{p-1}{h} \frac{\partial c_{1b_1 \dots mb_{m,i}}^{(h)} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{1e_1 \dots me_{m,j}}^{(p-1-h)} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \right\} \text{ and}$$

$$R_{1_{a_1} \dots m_{a_m}}^{N_{1_{a_1} \dots m_{a_m}}^{(p)}} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$$

$$= \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_1+\dots+d_m} v_{ij}(t_0, x_0) (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ \left. \frac{\partial \left[R_{01b_1 \dots mb_{m,i}} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \right]}{\partial x_i} \frac{\partial c_{1e_1 \dots me_{m,j}}^{(p-1)} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right].$$

If $k = 1$,

$$N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} =$$

$$\frac{1}{2} \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{b_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{d_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ \left. \frac{\partial \left[c_{0b_0 1_{b_1} \dots mb_{m,i}}^{(0)} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \right]}{\partial x_i} \frac{\partial \left[c_{0e_0 1_{e_1} \dots me_{m,j}}^{(0)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j}) \right]}{\partial x_j} \right]$$

$$= \frac{1}{2} \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{b_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{d_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ \left. \frac{\partial \left[-c_{0b_0+1 1_{b_1} \dots mb_{m,i}}^{(-1)} - \sum_{l=1}^{b_0} c_{0b_0-l 1_{b_1} \dots mb_{m,i}}^{(l)} \frac{1}{l!} + R_{0b_0+1 1_{b_1} \dots mb_{m,i}} \right] \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \times \right. \\ \left. \frac{\partial \left[-c_{0e_0+1 1_{e_1} \dots me_{m,j}}^{(-1)} - \sum_{k=1}^{e_0} c_{0e_0-k 1_{e_1} \dots me_{m,j}}^{(k)} \frac{1}{k!} + R_{0e_0+1 1_{e_1} \dots me_{m,j}} \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right]$$

$$= \frac{1}{2} \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{b_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{d_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ \left. \frac{\partial \left[c_{0b_0+1 1_{b_1} \dots mb_{m,i}}^{(-1)} \right] \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial \left[c_{0e_0+1 1_{e_1} \dots me_{m,j}}^{(-1)} \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right]$$

$$+ \frac{1}{2} \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{b_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{d_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right.$$

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$$\begin{aligned}
& \left[\frac{\partial \left[c_{0b_0+1}^{(-1)} 1_{b_1} \dots m_{b_m} i \right]}{\partial x_i} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \frac{\partial \left[\sum_{k=1}^{e_0} c_{0e_0-k}^{(k)} 1_{e_1} \dots m_{e_m} i \frac{1}{k!} \right]}{\partial x_j} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j}) \right] \\
& + \frac{1}{2} \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \quad \left. \frac{\partial \left[\sum_{l=1}^{b_0} c_{0b_0-l}^{(l)} 1_{b_1} \dots m_{b_m} i \frac{1}{l!} \right]}{\partial x_i} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \right. \\
& \quad \left. \frac{\partial \left[\sum_{k=1}^{e_0} c_{0e_0-k}^{(k)} 1_{e_1} \dots m_{e_m} i \frac{1}{k!} \right]}{\partial x_j} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j}) \right] \\
& - \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \quad \left. \frac{\partial \left[c_{0b_0+1}^{(-1)} 1_{b_1} \dots m_{b_m} i \right]}{\partial x_i} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \frac{\partial \left[R_{0e_0+1} 1_{e_1} \dots m_{e_m} j \right]}{\partial x_j} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j}) \right] \\
& - \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \quad \left. \frac{\partial \left[R_{0b_0+1} 1_{b_1} \dots m_{b_m} i \right]}{\partial x_i} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \frac{\partial \left[\sum_{k=1}^{e_0} c_{0e_0-k}^{(k)} 1_{e_1} \dots m_{e_m} i \frac{1}{k!} \right]}{\partial x_j} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j}) \right] \\
& + R \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
& = N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1,1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1,2)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + \\
& R N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + R \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}, \text{ i.e.}
\end{aligned}$$

$$N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1)} = N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1,1)} + N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1,2)} + R N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1)} + R. \quad (59)$$

$$\begin{aligned}
& \text{Here } N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1,1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
& = \frac{1}{2} \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \quad \left. \frac{\partial \left[c_{0b_0+1}^{(-1)} 1_{b_1} \dots m_{b_m} i \right]}{\partial x_i} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \frac{\partial \left[c_{0e_0+1}^{(-1)} 1_{e_1} \dots m_{e_m} j \right]}{\partial x_j} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j}) \right] \\
& = \frac{1}{2} \sum_{e_0=1}^p \sum_{b_0=1}^p \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \quad \left. \frac{\partial \left[c_{0b_0+1}^{(-1)} 1_{b_1} \dots m_{b_m} i \right]}{\partial x_i} \Delta^{b_0-1} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \frac{\partial \left[c_{0e_0}^{(-1)} 1_{e_1} \dots m_{e_m} j \right]}{\partial x_j} \Delta^{e_0-1} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j}) \right] \text{ and} \\
& R N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1)} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
& = - \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \quad \left. \frac{\partial \left[c_{0b_0+1}^{(-1)} 1_{b_1} \dots m_{b_m} i \right]}{\partial x_i} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i}) \frac{\partial \left[R_{0e_0+1} 1_{e_1} \dots m_{e_m} j \right]}{\partial x_j} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j}) \right]
\end{aligned}$$

$$- \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0}(x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ \left. \frac{\partial [R_{0, b_0+1^1 b_1 \dots m b_m i}] \Delta^{b_0}(x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial [\sum_{k=1}^{e_0} c_{0,0-k}^{(k)} e_1 \dots e_m i \frac{1}{k!}] \Delta^{e_0}(x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right].$$

$$\begin{aligned}
&= \sum_{k=1}^{p-1} \sum_{e_0=k}^{p-1} \sum_{b_0=1}^{p-k} \sum_{d_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial \left[\frac{c_{0b_0}^{(-1)} 1_{b_1} \dots m_{b_m} i}{\partial x_i} \Delta^{b_0-1} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i}) \right]}{\partial x_i} \frac{\partial \left[\frac{c_{0e_0-k}^{(k)} 1_{e_1} \dots m_{e_m} i}{\partial x_j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&+ \frac{1}{2} \sum_{k=1}^{p-2} \sum_{e_0=0}^{p-k-2} \sum_{b_0=1}^{p-k-1} \sum_{d_0=0}^{p-k-2} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial \left[\frac{\sum_{l=1}^{b_0} c_{0b_0-l}^{(l)} 1_{b_1} \dots m_{b_m} i}{\partial x_i} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i}) \right]}{\partial x_i} \frac{\partial \left[\frac{c_{0e_0}^{(k)} 1_{e_1} \dots m_{e_m} i}{\partial x_j} \Delta^{e_0+k} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&= \sum_{k=1}^{p-1} \sum_{e_0=0}^{p-k-1} \sum_{b_0=1}^{p-k} \sum_{d_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial \left[\frac{c_{0b_0}^{(-1)} 1_{b_1} \dots m_{b_m} i}{\partial x_i} \Delta^{b_0-1} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i}) \right]}{\partial x_i} \frac{\partial \left[\frac{c_{0e_0}^{(k)} 1_{e_1} \dots m_{e_m} i}{\partial x_j} \Delta^{e_0+k} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&+ \frac{1}{2} \sum_{k=1}^{p-2} \sum_{e_0=0}^{p-k-2} \sum_{b_0=1}^{p-k-1} \sum_{d_0=0}^{p-k-2} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial \left[\frac{\sum_{l=1}^{b_0} c_{0b_0-l}^{(l)} 1_{b_1} \dots m_{b_m} i}{\partial x_i} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i}) \right]}{\partial x_i} \frac{\partial \left[\frac{c_{0e_0}^{(k)} 1_{e_1} \dots m_{e_m} i}{\partial x_j} \Delta^{e_0+k} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&= - \sum_{k=2}^p N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,1)} \frac{1}{(k-1)!} \Delta^{p-1} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m}. \text{ Consequently,}
\end{aligned}$$

$$N_{0_{p-1} 1_{a_1} \dots m_{a_m}}^{(1,2)} = - \sum_{k=2}^p N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,1)} \frac{1}{(k-1)!}. \quad (60)$$

Let us show that

$$\sum_{k=2}^p N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,2)} \frac{1}{(k-1)!} = 0. \quad (61)$$

$$\begin{aligned}
&\sum_{k=2}^p N_{0_{p-k} 1_{a_1} \dots m_{a_m}}^{(k,2)} \Delta^{p-k} (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \frac{\Delta^k}{(k-1)!} \\
&= -\frac{1}{2} \sum_{k=2}^{p-1} \frac{\Delta^k}{(k-1)!} \sum_{e_0=0}^{p-k-1} \sum_{b_0=1}^{p-k} \sum_{d_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial \left[\frac{\sum_{h=1}^{b_0} c_{0b_0-h}^{(h)} 1_{b_1} \dots m_{b_m} i}{\partial x_i} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i}) \right]}{\partial x_i} \frac{\partial \left[\frac{c_{0e_0}^{(k-1)} 1_{e_1} \dots m_{e_m} j}{\partial x_j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&+ \frac{1}{2} \sum_{k=3}^p \frac{\Delta^k}{(k-1)!} \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left\{ \frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial \left[\frac{\sum_{h=1}^{k-2} \binom{k-1}{h} \frac{\partial c_{0b_0}^{(h)} 1_{b_1} \dots m_{b_m} i}{\partial x_i} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i}) \right]}{\partial x_i} \frac{\partial \left[\frac{c_{0e_0}^{(k-1-h)} 1_{e_1} \dots m_{e_m} j}{\partial x_j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right\} = \\
&0 \text{ because} \\
&-\frac{1}{2} \sum_{k=2}^{p-1} \frac{\Delta^k}{(k-1)!} \sum_{e_0=0}^{p-k-1} \sum_{b_0=1}^{p-k} \sum_{d_0=0}^{p-k-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right.
\end{aligned}$$

and

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$$\begin{aligned}
& \left[\sum_{h=1}^{k-2} \binom{k-1}{h} \frac{\partial c_{0b_0 1b_1 \dots m b_m i}^{(h)} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_0 1e_1 \dots m e_m j}^{(k-1-h)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \Bigg\} \\
&= \frac{1}{2} \sum_{k=3}^p \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left\{ \frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0b_0 1b_1 \dots m b_m i}^{(h)} \Delta^{b_0+k} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_0 1e_1 \dots m e_m j}^{(k-1-h)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right\} \\
&= \frac{1}{2} \sum_{h=1}^{k-2} \sum_{k=h+2}^p \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left\{ \frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0b_0 1b_1 \dots m b_m i}^{(h)} \Delta^{b_0+k} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_0 1e_1 \dots m e_m j}^{(k-1-h)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right\} \\
&= \left[\frac{1}{h!} \frac{\partial c_{0b_0 1b_1 \dots m b_m i}^{(h)} \Delta^{b_0+k} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{1}{(k-1-h)!} \frac{\partial c_{0e_0 1e_1 \dots m e_m j}^{(k-1-h)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \Bigg\}.
\end{aligned}$$

Let us see the $L_{0p 1a_1 \dots m a_m}^{(0)}$ term.

$$\begin{aligned}
& L_{0p 1a_1 \dots m a_m}^{(0)} \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} = \\
& \sum_{e_0=0}^p \sum_{b_0=0}^p \sum_{d_0=0}^p \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0b_0 1b_1 \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_0 1e_1 \dots m e_m j}^{(0)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \\
&= \sum_{e_0=0}^p \sum_{b_0=0}^p \sum_{d_0=0}^p \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0b_0 1b_1 \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial \left[-c_{0e_0+1 1e_1 \dots m e_m j}^{(-1)} - \sum_{k=1}^{e_0} c_{0e_0-k 1e_1 \dots m e_m j}^{(k)} + R_{0e_0+1 1e_1 \dots m e_m j} \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \\
&= - \sum_{e_0=0}^p \sum_{b_0=0}^p \sum_{d_0=0}^p \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0b_0 1b_1 \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial \left[c_{0e_0+1 1e_1 \dots m e_m j}^{(-1)} \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \\
&\quad - \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0b_0 1b_1 \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial \left[\sum_{k=1}^{e_0} c_{0e_0-k 1e_1 \dots m e_m j}^{(k)} \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right] \\
&\quad + \sum_{e_0=0}^p \sum_{b_0=0}^p \sum_{d_0=0}^p \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0b_0 1b_1 \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial \left[R_{0e_0+1 1e_1 \dots m e_m j} \right] \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right]
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{\substack{e_0=1 \\ e_0+b_0+d_0=p+1}}^{p+1} \sum_{b_0=0}^p \sum_{d_0=0}^p \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0_{e_0 1 e_1} \dots m e_m j}^{(-1)}}{\partial x_i} \frac{\partial \left[c_{0_{e_0 1 e_1} \dots m e_m j}^{(-1)} \Delta^{e_0-1} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&\quad \text{If } e_0=p+1 \text{ then } |b|>1 \text{ or } |d|>0 \\
&- \sum_{k=1}^p \sum_{e_0=k}^p \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0_{e_0 1 e_1} \dots m e_m j}^{(-1)}}{\partial x_i} \frac{\partial \left[c_{0_{e_0 1 e_1} \dots m e_m j}^{(-1)} \Delta^{e_0-1} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&\quad \text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0 \\
&\frac{\partial c_{0_{b_0 1 b_1} \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial \left[c_{0_{e_0 1 e_1} \dots m e_m j}^{(k)} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \\
&+ R^{L_{0p1a_1} \dots m a_m} \Delta^p (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \\
&= -\frac{1}{2} \sum_{e_0=1}^p \sum_{b_0=1}^p \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0_{b_0 1 b_1} \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial \left[c_{0_{e_0 1 e_1} \dots m e_m j}^{(-1)} \Delta^{e_0-1} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&\quad \text{If } e_0=p+1 \text{ then } |b|>1 \text{ or } |d|>0 \\
&- \frac{1}{2} \sum_{e_0=1}^p \sum_{b_0=1}^p \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0_{b_0 1 b_1} \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial \left[c_{0_{e_0 1 e_1} \dots m e_m j}^{(-1)} \Delta^{e_0-1} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&\quad \text{If } e_0=p+1 \text{ then } |b|>1 \text{ or } |d|>0 \\
&- \sum_{e_0=1}^{p+1} \sum_{d_0=0}^p \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \right. \\
&\quad \left. \frac{\partial c_{0_{b_0 1 b_1} \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial \left[c_{0_{e_0 1 e_1} \dots m e_m j}^{(-1)} \Delta^{e_0-1} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&\quad \text{If } e_0=p+1 \text{ then } |b|>1 \text{ or } |d|>0 \\
&- \sum_{k=1}^{p-1} L_{0_{p-k1a_1} \dots m a_m}^{(k)} \Delta^p (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \frac{1}{(k-1)!} \\
&+ R^{L_{0p1a_1} \dots m a_m} \Delta^p (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \\
&= -Q_{0_{p+11a_1} \dots m a_m}^{(-1)} \Delta^p (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} - N_{0_{p-11a_1} \dots m a_m}^{(1,1)} \Delta^p (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \\
&- \sum_{k=1}^{p-1} L_{0_{p-k1a_1} \dots m a_m}^{(k)} \Delta^p (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \frac{1}{(k-1)!} + R^{L_{0p1a_1} \dots m a_m} \Delta^p (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m}, \\
&\text{where } R^{L_{0p1a_1} \dots m a_m} \Delta^p (x_1-x_{01})^{a_1} \dots (x_m-x_{0m})^{a_m} \\
&= \sum_{e_0=0}^p \sum_{b_0=0}^p \sum_{d_0=0}^p \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
&\quad \left. \frac{\partial c_{0_{b_0 1 b_1} \dots m b_m i}^{(-1)} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial \left[R_{0_{e_0+11e_1} \dots m e_m j} \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j}) \right]}{\partial x_j} \right] \\
&\quad \text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0 \\
&= \sum_{e_0=0}^{p-1} \sum_{b_0=1}^p \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right.
\end{aligned}$$

$$\frac{\partial c_{0b_0+1b_1 \dots m_{b_m} i}^{(-1)} \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial [R_{0e_0+1e_1 \dots m_{e_m} j}] \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \Bigg] +$$

$$R \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}.$$

Accordingly,

$$L_{0p1a_1 \dots m_{a_m}}^{(0)} = -Q_{0p+11a_1 \dots m_{a_m}}^{(-1)} - N_{0p-11a_1 \dots m_{a_m}}^{(1,1)} - \sum_{k=1}^{p-1} L_{0p-k1a_1 \dots m_{a_m}}^{(k)} \frac{1}{(k-1)!} + R^{L_{0p1a_1 \dots m_{a_m}}^{(0)}} + R. \quad (62)$$

$$\begin{aligned} & \text{Notice that } R^{N_{0p-11a_1 \dots m_{a_m}}^{(1)}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \frac{\Delta}{1!} \\ &= - \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{b_0=0 \\ b_0+d_0=p-1}}^{p-1} \sum_{\substack{d_0=0 \\ d_0+p-1}}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ & \left. \frac{\partial [c_{0b_0+1b_1 \dots m_{b_m} i}^{(-1)}] \Delta^{b_0+1} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial [R_{0e_0+1e_1 \dots m_{e_m} j}] \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \\ & - \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{b_0=0 \\ b_0+d_0=p-1}}^{p-1} \sum_{\substack{d_0=0 \\ d_0+p-1}}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ & \left. \frac{\partial [R_{0b_0+1b_1 \dots m_{b_m} i}] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial [\sum_{k=1}^{e_0} c_{0e_0-k1e_1 \dots m_{e_m} i}^{(k)} \frac{1}{k!}] \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \frac{\Delta}{1!} \\ &= - \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p}}^{p-1} \sum_{\substack{b_0=1 \\ b_0+d_0=p}}^p \sum_{\substack{d_0=0 \\ d_0+p}}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ & \left. \frac{\partial [c_{0b_0+1b_1 \dots m_{b_m} i}^{(-1)}] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial [R_{0e_0+1e_1 \dots m_{e_m} j}] \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \\ & - \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{b_0=0 \\ b_0+d_0=p-1}}^{p-1} \sum_{\substack{d_0=0 \\ d_0+p-1}}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ & \left. \frac{\partial [R_{0b_0+1b_1 \dots m_{b_m} i}] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial [\sum_{k=1}^{e_0} c_{0e_0-k1e_1 \dots m_{e_m} i}^{(k)} \frac{1}{k!}] \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \frac{\Delta}{1!}. \end{aligned}$$

First two lines of the last equation are equal to $-R^{L_{0p1a_1 \dots m_{a_m}}^{(0)}} \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m}$ and last

two lines of the last equation can be written as

$$\begin{aligned} & - \sum_{k=1}^{p-1} \sum_{\substack{e_0=k \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{\substack{b_0=0 \\ b_0+d_0=p-1}}^{p-k-1} \sum_{\substack{d_0=0 \\ d_0+p-1}}^{p-k-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ & \left. \frac{\partial [R_{0b_0+1b_1 \dots m_{b_m} i}] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial [c_{0e_0-k1e_1 \dots m_{e_m} i}^{(k)} \frac{1}{k!}] \Delta^{e_0} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \frac{\Delta}{1!} (k) \\ &= - \sum_{k=1}^{p-1} \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-k-1}}^{p-k-1} \sum_{\substack{b_0=0 \\ b_0+d_0=p-k-1}}^{p-k-1} \sum_{\substack{d_0=0 \\ d_0+p-k-1}}^{p-k-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{|d|=0 \\ e+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \dots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\ & \left. \frac{\partial [R_{0b_0+1b_1 \dots m_{b_m} i}] \Delta^{b_0} (x_1-x_{01})^{b_1} \dots (x_m-x_{0m})^{b_m} (x_i-x_{0i})}{\partial x_i} \frac{\partial [c_{0e_0+1e_1 \dots m_{e_m} i}^{(k)} \frac{1}{k!}] \Delta^{e_0+k} (x_1-x_{01})^{e_1} \dots (x_m-x_{0m})^{e_m} (x_j-x_{0j})}{\partial x_j} \right] \frac{\Delta}{1!} \\ &= - \sum_{k=2}^p R^{N_{0p-k1a_1 \dots m_{a_m}}^{(k)}} \frac{1}{(k-1)!} \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \text{ because, for } 2 \leq k \leq p, \end{aligned}$$

$$\begin{aligned}
& R^{N_{0_{p-k}1_{a_1}\dots m_{a_m}}} \Delta^{p-k} (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \\
&= \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-k}}^{p-k} \sum_{\substack{b_0=0 \\ b_0+d_0=p-k}}^{p-k} \sum_{\substack{d_0=0 \\ d_0+b_0+d_0=p-k}}^{p-k} \sum_{\substack{e=0 \\ e+b+d=a}}^{n+1} \sum_{\substack{b=0 \\ b+d=a}}^{n+1} \sum_{\substack{d=0 \\ d+b+d=a}}^{n+1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0+d_1+\dots+d_m} v_{ij}(t_0, x_0) \Delta^{d_0} (x_1 - x_{01})^{d_1} \dots (x_m - x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \dots \partial x_m^{d_m}} \times \right. \\
& \left. \frac{\partial [R_{0_{b_0+1}1_{b_1}\dots m_{b_m}}] \Delta^{b_0} (x_1 - x_{01})^{b_1} \dots (x_m - x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0}1_{e_1}\dots m_{e_m}}^{(k-1)} \Delta^{e_0} (x_1 - x_{01})^{e_1} \dots (x_m - x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right].
\end{aligned}$$

For these reasons, $R^{L_{0_{p-1}1_{a_1}\dots m_{a_m}}} \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} =$

$$- \sum_{k=1}^p R^{N_{0_{p-k}1_{a_1}\dots m_{a_m}}} \frac{1}{(k-1)!} \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} + R \Delta^p (x_1 - x_{01})^{a_1} \dots (x_m - x_{0m})^{a_m} \text{ and}$$

$$R^{L_{0_{p-1}1_{a_1}\dots m_{a_m}}} = - \sum_{k=1}^p R^{N_{0_{p-k}1_{a_1}\dots m_{a_m}}} \frac{1}{(k-1)!} + R. \quad (63)$$

Making use of (53), (54), and (55), $c_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)}$, for $1 \leq k \leq p-1$ and $c_{1_{a_1}\dots m_{a_m}}^{(p)}$ can be expressed in terms of $c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)}$, L , P , and N as follows.

$$\begin{aligned}
& \text{When } 1 \leq k \leq p-1, \quad c_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \\
&= \frac{k!|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^{k+1} \frac{(p+1)!}{(p-k)!} c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \\
&+ \frac{k!|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^k \frac{p!}{(p-k)!} \left(L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(0)} + P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(0)} \right) \\
&+ \frac{k!|a|!}{(|a|+k)!} (-1)^{k-1} \frac{(p-1)!}{(p-k)!} \left(L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} \right) \\
&+ \frac{k!|a|+1!}{1!|a|+k)!} (-1)^{k-2} \frac{(p-2)!}{(p-k)!} \left(L_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + P_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} \right) \\
&+ \frac{k!|a|+2!}{2!|a|+k)!} (-1)^{k-3} \frac{(p-3)!}{(p-k)!} \left(L_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} + P_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} + N_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} \right) \\
&+ \dots + \\
&+ \frac{k!|a|+k-2!}{(k-2)!|a|+k)!} (-1)^{p-k+1} \frac{(p-k+1)!}{(p-k)!} \left(L_{0_{p-k+1}1_{a_1}\dots m_{a_m}}^{(k-1)} + P_{0_{p-k+1}1_{a_1}\dots m_{a_m}}^{(k-1)} + N_{0_{p-k+1}1_{a_1}\dots m_{a_m}}^{(k-1)} \right) \\
&+ \frac{k!|a|+k-1!}{(k-1)!|a|+k)!} (-1)^0 \frac{(p-k)!}{(p-k)!} \left(L_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} + P_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} + N_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \right) \text{ and if } k = p, \\
&c_{1_{a_1}\dots m_{a_m}}^{(p)} \\
&= \frac{p!|a|!}{(|a|+p)!} \frac{1}{|a|} (-1)^{p+1} \frac{(p+1)!}{0!} c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \\
&+ \frac{p!|a|!}{(|a|+p)!} \frac{1}{|a|} (-1)^p \frac{p!}{0!} \left(L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(0)} + P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(0)} \right) \\
&+ \frac{p!|a|!}{(|a|+p)!} (-1)^{p-1} \frac{(p-1)!}{0!} \left(L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} \right) \\
&+ \frac{p!|a|+1!}{1!|a|+p)!} (-1)^{p-2} \frac{(p-2)!}{0!} \left(L_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + P_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} \right) \\
&+ \frac{p!|a|+2!}{2!|a|+p)!} (-1)^{p-3} \frac{(p-3)!}{0!} \left(L_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} + P_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} + N_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} \right) \\
&+ \dots + \\
&+ \frac{p!|a|+p-2!}{(p-2)!|a|+p)!} (-1)^1 \frac{1!}{0!} \left(L_{0_{1a_1}\dots m_{a_m}}^{(p-1)} + P_{0_{1a_1}\dots m_{a_m}}^{(p-1)} + N_{0_{1a_1}\dots m_{a_m}}^{(p-1)} \right) \\
&+ \frac{p!|a|+p-1!}{(p-1)!|a|+p)!} \left(P_{1_{a_1}\dots m_{a_m}}^{(p)} + N_{1_{a_1}\dots m_{a_m}}^{(p)} \right).
\end{aligned}$$

In $c_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)}$ and $c_{1_{a_1}\dots m_{a_m}}^{(p)}$, if we replace $P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(0)}$ with $-\sum_{k=1}^p P_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \frac{1}{(k-1)!}$ due to (56), $L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(0)}$ with the right-hand side of (62) after putting $(|a|-1)c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} + R$ in place of $Q_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)}$ because of (52), $N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)}$ with the right-hand side of (59) after substituting $-\sum_{k=2}^p N_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k,1)} \frac{1}{(k-1)!}$ for $N_{0_{p-1}1_{a_1}\dots m_{a_m}}^{(1,2)}$ as a result from (60), $N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)}$ with the right-hand side of

(57) for $k = 2$ after substituting $N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2,2)}$ by $\sum_{l=3}^p N_{0_{p-l}1_{a_1}\dots m_{a_m}}^{(l,2)} \frac{1}{(l-1)!}$ thanks to (61), and $N_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)}$ with the right-hand side of (57) for $3 \leq k \leq p$, then we get, for $1 \leq k \leq p-1$,

$$\begin{aligned}
& c_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \frac{1}{k!} \\
&= \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^{k+1} \frac{(p+1)!}{(p-k)!} c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \\
&+ \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^k \frac{p!}{(p-k)!} \left[-(|a|-1) c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} - N_{0_{p-1}1_{a_1}\dots m_{a_m}}^{(1,1)} - \sum_{k=1}^{p-1} L_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \frac{1}{(k-1)!} + \right. \\
&\quad \left. R_{0_{p-1}1_{a_1}\dots m_{a_m}}^{L(0)} - \sum_{k=1}^p P_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \frac{1}{(k-1)!} \right] \\
&+ \frac{|a|!}{(|a|+k)!} (-1)^{k-1} \frac{(p-1)!}{(p-k)!} \left[L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1,1)} - \right. \\
&\quad \left. \sum_{i=2}^p N_{0_{p-i}1_{a_1}2_{a_2}\dots m_{a_m}}^{(i,1)} \frac{1}{(i-1)!} + R_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{N(1)} \right] \\
&+ \frac{(|a|+1)!}{1!(|a|+k)!} (-1)^{k-2} \frac{(p-2)!}{(p-k)!} \left[L_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + P_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2,1)} - \right. \\
&\quad \left. \sum_{l=3}^p N_{0_{p-l}1_{a_1}\dots m_{a_m}}^{(l,2)} \frac{1}{(l-1)!} + R_{0_{p-2}1_{a_1}\dots m_{a_m}}^{N(2)} \right] \\
&+ \frac{(|a|+2)!}{2!(|a|+k)!} (-1)^{k-3} \frac{(p-3)!}{(p-k)!} \left[L_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} + P_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} + N_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3,1)} + \right. \\
&\quad \left. N_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3,2)} + R_{0_{p-3}1_{a_1}\dots m_{a_m}}^{N(3)} \right] + \dots \\
&+ \frac{(|a|+k-2)!}{(k-2)!(|a|+k)!} (-1)^{p-k+1} \frac{(p-k+1)!}{(p-k)!} \left[L_{0_{p-k+1}1_{a_1}\dots m_{a_m}}^{(k-1)} + P_{0_{p-k+1}1_{a_1}\dots m_{a_m}}^{(k-1)} + N_{0_{p-k+1}1_{a_1}\dots m_{a_m}}^{(k-1,1)} + \right. \\
&\quad \left. N_{0_{p-k+1}1_{a_1}\dots m_{a_m}}^{(k-1,2)} + R_{0_{p-k+1}1_{a_1}\dots m_{a_m}}^{N(k-1)} \right] \\
&+ \frac{(|a|+k-1)!}{(k-1)!(|a|+k)!} (-1)^0 \frac{(p-k)!}{(p-k)!} \left[L_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} + P_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} + N_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k,1)} + \right. \\
&\quad \left. N_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k,2)} + R_{0_{p-k}1_{a_1}\dots m_{a_m}}^{N(k)} \right] \text{ and for } k = p, \\
&c_{1_{a_1}\dots m_{a_m}}^{(p)} \frac{1}{p!} = \frac{|a|!}{(|a|+p)!} \frac{1}{|a|} (-1)^{p+1} \frac{(p+1)!}{0!} c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \\
&+ \frac{|a|!}{(|a|+p)!} \frac{1}{|a|} (-1)^p \frac{p!}{0!} \left[-(|a|-1) c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} - N_{0_{p-1}1_{a_1}\dots m_{a_m}}^{(1,1)} - \sum_{k=1}^{p-1} L_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \frac{1}{(k-1)!} + \right. \\
&\quad \left. R_{0_{p-1}1_{a_1}\dots m_{a_m}}^{L(0)} - \sum_{k=1}^p P_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \frac{1}{(k-1)!} \right] \\
&+ \frac{|a|!}{(|a|+p)!} (-1)^{p-1} \frac{(p-1)!}{0!} \left[L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1,1)} - \right. \\
&\quad \left. \sum_{i=2}^p N_{0_{p-i}1_{a_1}2_{a_2}\dots m_{a_m}}^{(i,1)} \frac{1}{(i-1)!} + R_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{N(1)} \right] \\
&+ \frac{(|a|+1)!}{1!(|a|+p)!} (-1)^{p-2} \frac{(p-2)!}{0!} \left[L_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + P_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2,1)} - \right. \\
&\quad \left. \sum_{l=3}^p N_{0_{p-l}1_{a_1}\dots m_{a_m}}^{(l,2)} \frac{1}{(l-1)!} + R_{0_{p-2}1_{a_1}\dots m_{a_m}}^{N(2)} \right] \\
&+ \frac{(|a|+2)!}{2!(|a|+p)!} (-1)^{p-3} \frac{(p-3)!}{0!} \left[L_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} + P_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3)} + N_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3,1)} + \right. \\
&\quad \left. N_{0_{p-3}1_{a_1}\dots m_{a_m}}^{(3,2)} + R_{0_{p-3}1_{a_1}\dots m_{a_m}}^{N(3)} \right] + \dots + \\
&+ \frac{(|a|+p-2)!}{(p-2)!(|a|+p)!} (-1)^1 \frac{1!}{0!} \left(L_{01_{a_1}\dots m_{a_m}}^{(p-1)} + P_{01_{a_1}\dots m_{a_m}}^{(p-1)} + N_{01_{a_1}\dots m_{a_m}}^{(p-1,1)} + N_{01_{a_1}\dots m_{a_m}}^{(p-1,2)} + R_{01_{a_1}\dots m_{a_m}}^{N(p-1)} \right) \\
&+ \frac{(|a|+p-1)!}{(p-1)!(|a|+p)!} \left(P_{1_{a_1}\dots m_{a_m}}^{(p)} + N_{1_{a_1}\dots m_{a_m}}^{(p,1)} + N_{1_{a_1}\dots m_{a_m}}^{(p,2)} + R_{1_{a_1}\dots m_{a_m}}^{N(p)} \right).
\end{aligned}$$

As a result,

$$\sum_{k=-1}^p c_{0_{p-k}1_{a_1}\dots m_{a_m}}^{(k)} \frac{1}{k!}$$

$$\begin{aligned}
&= c_{0_{p+1}1_{a_1}\dots m_{a_m}}^{(-1)} \\
&+ \sum_{k=0}^p \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^{k+1} \frac{p!(p+|a|)}{(p-k)!} c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \\
&+ \sum_{k=0}^p \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^k \frac{p!}{(p-k)!} \left[-N_{0_{p-1}1_{a_1}\dots m_{a_m}}^{(1,1)} - \sum_{l=1}^{p-1} L_{0_{p-l}1_{a_1}\dots m_{a_m}}^{(l)} \frac{1}{(l-1)!} - \sum_{j=1}^p R^{N_{0_{p-j}1_{a_1}\dots m_{a_m}}^{(j)}} \frac{1}{(j-1)!} - \right. \\
&\quad \left. \sum_{h=1}^p P_{0_{p-h}1_{a_1}\dots m_{a_m}}^{(h)} \frac{1}{(h-1)!} \right] \\
&+ \sum_{k=1}^p \frac{|a|!}{(|a|+k)!} (-1)^{k-1} \frac{(p-1)!}{(p-k)!} \left[L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1,1)} - \right. \\
&\quad \left. \sum_{i=2}^p N_{0_{p-i}1_{a_1}2_{a_2}\dots m_{a_m}}^{(i,1)} \frac{1}{(i-1)!} + R^{N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)}} \right] \\
&+ \sum_{k=2}^p \frac{(|a|+1)!}{1!(|a|+k)!} (-1)^{k-2} \frac{(p-2)!}{(p-k)!} \left[L_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + P_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2,1)} - \right. \\
&\quad \left. \sum_{l=3}^p N_{0_{p-l}1_{a_1}\dots m_{a_m}}^{(l,2)} \frac{1}{(l-1)!} + R^{N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)}} \right] \\
&+ \sum_{i=3}^{p-1} \sum_{k=i}^p \frac{(|a|+i-1)!}{(i-1)!(|a|+k)!} (-1)^{k-i} \frac{(p-i)!}{(p-k)!} \left[L_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i)} + P_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i)} + \right. \\
&\quad \left. N_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i,1)} + N_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i,2)} + R^{N_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i)}} \right] \\
&+ \dots \\
&+ \sum_{k=p}^p \frac{(|a|+p-1)!}{(p-1)!(|a|+k)!} (-1)^{k-p} \frac{(p-p)!}{(p-k)!} \left(P_{1_{a_1}\dots m_{a_m}}^{(p)} + N_{1_{a_1}\dots m_{a_m}}^{(p,1)} + N_{1_{a_1}\dots m_{a_m}}^{(p,2)} + R^{N_{1_{a_1}\dots m_{a_m}}^{(p)}} \right) + R \\
&= \left(1 + \sum_{k=0}^p \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^{k+1} \frac{p!(p+|a|)}{(p-k)!} \right) c_{0_{p+1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(-1)} \text{ or} \\
&+ \left(L_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + P_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)} + N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1,1)} + R^{N_{0_{p-1}1_{a_1}2_{a_2}\dots m_{a_m}}^{(1)}} \right) \times \\
&\quad \left[\sum_{k=0}^p \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^{k+1} \frac{p!}{(p-k)!} - \sum_{k=1}^p \frac{|a|!}{(|a|+k)!} (-1)^k \frac{(p-1)!}{(p-k)!} \right] \\
&+ \left(L_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + P_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)} + R^{N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2)}} \right) \times \\
&\quad \left[\sum_{k=0}^p \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^{k+1} \frac{p!}{(p-k)!} - \sum_{k=2}^p \frac{(|a|+1)!}{1!(|a|+k)!} (-1)^{k-1} \frac{(p-2)!}{(p-k)!} \right] \\
&+ N_{0_{p-2}1_{a_1}\dots m_{a_m}}^{(2,1)} \left[\sum_{k=1}^p \frac{|a|!}{(|a|+k)!} (-1)^k \frac{(p-1)!}{(p-k)!} - \sum_{k=2}^p \frac{(|a|+1)!}{1!(|a|+k)!} (-1)^{k-1} \frac{(p-2)!}{(p-k)!} \right] \\
&+ \sum_{i=3}^{p-1} \left(L_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i)} + P_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i)} + R^{N_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i)}} \right) \times \\
&\quad \left[\sum_{k=0}^p \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^{k+1} \frac{p!}{(p-k)!} \frac{1}{(i-1)!} - \sum_{k=i}^p \frac{(|a|+i-1)!}{(i-1)!(|a|+k)!} (-1)^{k+1-i} \frac{(p-i)!}{(p-k)!} \right] \\
&+ \sum_{i=3}^p N_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i,1)} \left[\sum_{k=1}^p \frac{|a|!}{(|a|+k)!} (-1)^k \frac{(p-1)!}{(p-k)!} \frac{1}{(i-1)!} - \sum_{k=i}^p \frac{(|a|+i-1)!}{(i-1)!(|a|+k)!} (-1)^{k+1-i} \frac{(p-i)!}{(p-k)!} \right] \\
&+ \sum_{i=3}^p N_{0_{p-i}1_{a_1}\dots m_{a_m}}^{(i,2)} \left[\sum_{k=2}^p \frac{(|a|+1)!}{1!(|a|+k)!} (-1)^{k-1} \frac{(p-2)!}{(p-k)!} \frac{1}{(i-1)!} - \sum_{k=i}^p \frac{(|a|+i-1)!}{(i-1)!(|a|+k)!} (-1)^{k+1-i} \frac{(p-i)!}{(p-k)!} \right] \\
&+ \left(P_{1_{a_1}\dots m_{a_m}}^{(p)} + R^{N_{1_{a_1}\dots m_{a_m}}^{(p)}} \right) \left[\sum_{k=0}^p \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^{k+1} \frac{p!}{(p-k)!} \frac{1}{(i-1)!} - \sum_{k=p}^p \frac{(|a|+p-1)!}{(p-1)!(|a|+k)!} (-1)^{k+1-p} \frac{(p-p)!}{(p-k)!} \right] + \\
&R
\end{aligned}$$

R

$= R_{0_{p+1}1_{a_1}\dots m_{a_m}}$ since

$$\begin{aligned}
&\sum_{k=i}^p \frac{(|a|+i-1)!}{(i-1)!(|a|+k)!} (-1)^{k+1-i} \frac{(p-i)!}{(p-k)!} = -\frac{1}{p+|a|} \frac{1}{(i-1)!} \text{ for all } i, 1 \leq i \leq p \text{ and a non-negative integer } |a| \text{ be-} \\
&\text{cause of Lemma A4 and the fact that } \sum_{k=0}^p \frac{|a|!}{(|a|+k)!} \frac{1}{|a|} (-1)^{k+1} \frac{p!}{(p-k)!} = -\frac{1}{|a|} - p \frac{1}{|a|} \sum_{k=1}^p \frac{|a|!}{(|a|+k)!} (-1)^k \frac{(p-1)!}{(p-k)!} \\
&= -\frac{1}{|a|} - \frac{p}{|a|} \left(-\frac{1}{p+|a|} \right) = -\frac{1}{|a|} \left[1 - \frac{p}{p+|a|} \right] = -\frac{1}{|a|} \left[\frac{|a|}{p+|a|} \right] = -\frac{1}{p+|a|}. \blacksquare
\end{aligned}$$

Lemma A6 The coefficient of Δ^{a_0} in $\tilde{l}_X^{(\infty)}(t, x \mid t_0, x_0)$ is free from the indeterminate terms of $C_X^{(j-1, -1)}$ for all $a_0 \geq 1$.

Proof. As can be seen from the proof of Lemma A1, the $c_{0p}^{(0)}(t_0, x_0)$, $p \geq 1$, terms are cancelled out by $C_X^{(jk,k)}$, $1 \leq k \leq p$ in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$. Hence, we can focus on the coefficients of Δ^p , $p \geq 1$, that are dependent on the indeterminate terms of $C_X^{(j-1,-1)}$ in each $C_X^{(jk,k)}$, $k \neq 0$, in order to prove this theorem.

First consider the case $a_0 = 1$. The Δ term of $\tilde{l}_X^{(\infty)}$ comes from $\frac{c_{02}^{(-1)}\Delta^2}{\Delta} + c^{(1)}\Delta$, which can be shown to not have indeterminacy problem analogously to the proof below and details are excluded. One difference is that $N^{(1,2)}$ term is not associated in this case.

Now consider the coefficient of Δ^p , $p \geq 2$ in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$. This term is obtained from $\frac{c_{0p+1}^{(-1)}\Delta^{p+1}}{\Delta} + \sum_{k=1}^p c_{0p-k}^{(k)}\Delta^{p-k}\frac{\Delta^k}{k!}$.

Using $\tilde{f}_X^{(-2)}(t, x|t_0, x_0) = 0$, $c_{0p+1}^{(-1)}$ satisfies $-2c_{0p+1}^{(-1)}\Delta^{p+1} = 2Q_{0p+1}^{(-1)}\Delta^{p+1}$, where

$$Q_{0p+1}^{(-1)}\Delta^{p+1} = \frac{1}{2} \sum_{e_0=1}^p \sum_{b_0=1}^p \sum_{d_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{d_0} v_{ij}(t_0, x_0)}{\partial t^{d_0}} \frac{\partial c_{0b_0}^{(-1)} \Delta^{b_0} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_0}^{(-1)} \Delta^{e_0} (x_j - x_{0j})}{\partial x_j} \text{ so}$$

$$-Q_{0p+1}^{(-1)} = c_{0p+1}^{(-1)}. \quad (64)$$

The Δ^p term in $\tilde{f}_X^{(-1)}(t, x|t_0, x_0) = 0$ is $-L_{0p}^{(0)}\Delta^p = -\frac{\partial c_{0p+1}^{(-1)}\Delta^{p+1}}{\partial t} + P_{0p}^{(0)}\Delta^p + R\Delta^p$, where

$$L_{0p}^{(0)}\Delta^p = \sum_{e_0=0}^{p-1} \sum_{b_0=1}^p \sum_{d_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{d_0} v_{ij}(t_0, x_0)}{\partial t^{d_0}} \frac{\partial c_{0b_0}^{(-1)} \Delta^{b_0} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_0}^{(0)} \Delta^{e_0} (x_j - x_{0j})}{\partial x_j} \text{ and}$$

$$P_{0p}^{(0)}\Delta^p = - \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} \mu_i(t_0, x_0)}{\partial t^{b_0}} \frac{\partial c_{0e_0}^{(-1)} \Delta^{e_0} (x_i - x_{0i})}{\partial x_i} + \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} v_{ij}(t_0, x_0)}{\partial x_i^{b_0}} \frac{\partial c_{0e_0}^{(-1)} \Delta^{e_0} (x_j - x_{0j})}{\partial x_j}$$

$$- \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} v_{ij}(t_0, x_0)}{\partial t^{b_0}} \frac{\partial c_{0e_0}^{(-1)} \Delta^{e_0} (x_i - x_{0i})}{\partial x_i} \frac{\partial^{d_0} \left[\frac{\partial D_{ij}(t_0, x_0)}{\partial x_j} \right] \Delta^{d_0}}{\partial t^{d_0}} +$$

$$\frac{1}{2} \sum_{e_0=1}^p \sum_{b_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} v_{ij}(t_0, x_0)}{\partial t^{b_0}} \frac{\partial c_{0e_0}^{(-1)} \Delta^{e_0} (x_i - x_{0i}) (x_j - x_{0j})}{\partial x_i \partial x_j}. \text{ Consequently}$$

$$L_{0p}^{(0)} = (p+1)c_{0p+1}^{(-1)} - P_{0p}^{(0)} + R. \quad (65)$$

Recall that we can ignore $c_{0p}^{(0)}$ because of Lemma A1. The equation for $c_{0p-1}^{(1)}$ can be found from $\tilde{f}_X^{(0)}(t, x|t_0, x_0) = 0$ as

$c_{0p-1}^{(1)}\Delta^{p-1} - L_{0p-1}^{(1)}\Delta^{p-1} = P_{0p-1}^{(1)}\Delta^{p-1} + N_{0p-1}^{(1)}\Delta^{p-1} + R\Delta^{p-1}$. Here

$$L_{0p-1}^{(1)}\Delta^{p-1} = \sum_{e_0=0}^{p-2} \sum_{b_0=1}^{p-1} \sum_{d_0=0}^{p-2} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{d_0} v_{ij}(t_0, x_0)}{\partial t^{d_0}} \frac{\partial c_{0b_0}^{(-1)} \Delta^{b_0} (x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0e_0}^{(1)} \Delta^{e_0} (x_j - x_{0j})}{\partial x_j},$$

$$P_{0p-1}^{(1)}\Delta^{p-1} = - \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} \mu_i(t_0, x_0)}{\partial t^{b_0}} \frac{\partial c_{0e_0}^{(0)} \Delta^{e_0} (x_i - x_{0i})}{\partial x_i} + \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} v_{ij}(t_0, x_0)}{\partial x_i^{b_0}} \frac{\partial c_{0e_0}^{(0)} \Delta^{e_0} (x_j - x_{0j})}{\partial x_j}$$

$$- \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} v_{ij}(t_0, x_0)}{\partial t^{b_0}} \frac{\partial c_{0e_0}^{(0)} \Delta^{e_0} (x_i - x_{0i})}{\partial x_i} \frac{\partial^{d_0} \left[\frac{\partial D_{ij}(t_0, x_0)}{\partial x_j} \right] \Delta^{d_0}}{\partial t^{d_0}} +$$

$$\frac{1}{2} \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} v_{ij}(t_0, x_0)}{\partial t^{b_0}} \frac{\partial c_{0e_0}^{(0)} \Delta^{e_0} (x_i - x_{0i}) (x_j - x_{0j})}{\partial x_i \partial x_j} \text{ and}$$

$N_{0_{p-1}}^{(1)} \Delta^{p-1} = \frac{1}{2} \sum_{e_0=0}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{d_0} v_{ij}(t_0, x_0)}{\partial t^{d_0}} \frac{\partial c_{0_{b_0} i}^{(0)} \Delta^{b_0}(x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} j}^{(0)} \Delta^{e_0}(x_j - x_{0j})}{\partial x_j}$. Therefore

$$c_{0_{p-1}}^{(1)} - L_{0_{p-1}}^{(1)} = P_{0_{p-1}}^{(1)} + N_{0_{p-1}}^{(1)} + R. \quad (66)$$

Consider the term Δ^{p-k} in $\tilde{f}_X^{(k-1)}(t, x | t_0, x_0) = 0$, $2 \leq k \leq p$, then

$$c_{0_{p-k}}^{(k)} \Delta^{p-k} - L_{0_{p-k}}^{(k)} \Delta^{p-k} = -\frac{\partial c_{0_{p-k+1}}^{(k-1)} \Delta^{p-k+1}}{\partial t} + P_{0_{p-k}}^{(k)} \Delta^{p-k} + N_{0_{p-k}}^{(k)} \Delta^{p-k} + R \Delta^{p-k}. \text{ Here}$$

$$L_{0_{p-k}}^{(k)} \Delta^{p-k} = \frac{1}{k} \sum_{e_0=0}^{p-k-1} \sum_{b_0=1}^{p-k} \sum_{d_0=0}^{p-k-1} \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\partial^{d_0} v_{ij}(t_0, x_0) \Delta^{d_0}}{\partial t^{d_0}} \frac{\partial c_{0_{b_0} i}^{(-1)} \Delta^{b_0}(x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} j}^{(k)} \Delta^{e_0}(x_j - x_{0j})}{\partial x_j} \right].$$

$$\text{Note that } P_{0_{p-k}}^{(k)} \Delta^{p-k} = \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{i=1}^m \frac{\partial^{b_0} \mu_i(t_0, x_0) \Delta^{b_0}}{\partial t^{b_0}} \frac{\partial c_{0_{e_0} i}^{(k-1)} \Delta^{e_0}(x_i - x_{0i})}{\partial x_i} +$$

$$\sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0}}{\partial t^{b_0}} \left[\frac{\partial v_{ij}(t_0, x_0)}{\partial x_i} \right] \Delta^{b_0} \frac{\partial c_{0_{e_0} j}^{(k-1)} \Delta^{e_0}(x_j - x_{0j})}{\partial x_j}$$

$$- \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} v_{ij}(t_0, x_0) \Delta^{b_0}}{\partial t^{b_0}} \frac{\partial c_{0_{e_0} i}^{(k-1)} \Delta^{e_0}(x_i - x_{0i})}{\partial x_i} \frac{\partial^{d_0}}{\partial t^{d_0}} \left[\frac{\partial D_v(t_0, x_0)}{\partial x_j} \right] \Delta^{d_0}$$

$$+ \frac{1}{2} \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{b_0} v_{ij}(t_0, x_0) \Delta^{b_0}}{\partial t^{b_0}} \frac{\partial c_{0_{e_0} ij}^{(k-1)} \Delta^{e_0}(x_i - x_{0i})(x_j - x_{0j})}{\partial x_i \partial x_j} \text{ and}$$

$$N_{0_{p-k}}^{(k)} \Delta^{p-k} =$$

$$\sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{d_0} v_{ij}(t_0, x_0) \Delta^{d_0}}{\partial t^{d_0}} \frac{\partial c_{0_{b_0} i}^{(0)} \Delta^{b_0}(x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} j}^{(k-1)} \Delta^{e_0}(x_j - x_{0j})}{\partial x_j}$$

$$+ \frac{1}{2} \sum_{e_0=0}^{p-k} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^{d_0} v_{ij}(t_0, x_0) \Delta^{d_0}}{\partial t^{d_0}} \left[\sum_{h=1}^{k-2} \binom{k-1}{h} \frac{\partial c_{0_{b_0} i}^{(h)} \Delta^{b_0}(x_i - x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0} j}^{(k-1-h)} \Delta^{e_0}(x_j - x_{0j})}{\partial x_j} \right]$$

that are the Δ^{p-k} terms from $\tilde{G}_X^{(k,1)}(t, x | t_0, x_0)$ and $\tilde{G}_X^{(k,2)}(t, x | t_0, x_0)$, respectively, for $k \geq 2$. Accordingly,

$$c_{0_{p-k}}^{(k)} - L_{0_{p-k}}^{(k)} = -(p-k+1) c_{0_{p-k+1}}^{(k-1)} + P_{0_{p-k}}^{(k)} + N_{0_{p-k}}^{(k)} + R. \quad (67)$$

Using the fact that $c_{0_{e_0} i}^{(-1)} = -\sum_{k=0}^{e_0-1} c_{0_{e_0-k-1} i}^{(k)} \frac{1}{k!} + R_{0_{e_0} i}$ and $c_{0_{b_0} i}^{(0)} = -c_{0_{b_0+1} i}^{(-1)} - \sum_{l=1}^{b_0} c_{0_{b_0-l} i}^{(l)} \frac{1}{l!} + R_{0_{b_0+1} i}$ from Lemma A2, A3 and A5, similarly to the proof of Lemma A5, we get

$$P_{0_p}^{(0)} = -\sum_{k=1}^p P_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} + R, \quad (68)$$

for $1 \leq k \leq p$,

$$N_{0_{p-k}}^{(k)} = N_{0_{p-k}}^{(k,1)} + N_{0_{p-k}}^{(k,2)} + R^{N_{0_{p-k}}^{(k)}} + R, \quad (69)$$

$$N_{0_{p-1}}^{(1,2)} = -\sum_{k=2}^p N_{0_{p-k}}^{(k,1)} \frac{1}{(k-1)!} + R, \quad (70)$$

$$\sum_{k=2}^p N_{0_{p-k}}^{(k,2)} \frac{1}{(k-1)!} = 0, \quad (71)$$

$$L_{0_p}^{(0)} = -Q_{0_{p+1}}^{(-1)} - N_{0_{p-1}}^{(1,1)} - \sum_{k=1}^{p-1} L_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} + R^{L_{0_p}^{(0)}}. \quad (72)$$

and

$$R^{L_{0_p}^{(0)}} = - \sum_{k=1}^p R^{N_{0_{p-k}}^{(k)}} \frac{1}{(k-1)!} + R. \quad (73)$$

We are ready to show that $c_{0_{p+1}}^{(-1)} + \sum_{k=1}^p c_{0_{p-k}}^{(k)} \frac{1}{k!} = R_{0_{p+1}}$. First, using (64) and (73), (72) becomes

$$L_{0_p}^{(0)} = c_{0_{p+1}}^{(-1)} - N_{0_{p-1}}^{(1,1)} - \sum_{k=1}^{p-1} L_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} - \sum_{k=1}^p R^{N_{0_{p-k}}^{(k)}} \frac{1}{(k-1)!}$$

and (65) is equivalent to

$$L_{0_p}^{(0)} = (p+1) c_{0_{p+1}}^{(-1)} + \sum_{k=1}^p P_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} + R$$

due to (68). Combining these two

$$c_{0_{p+1}}^{(-1)} = \frac{1}{p} \left[- \sum_{k=1}^p P_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} - N_{0_{p-1}}^{(1,1)} - \sum_{k=1}^{p-1} L_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} - \sum_{k=1}^p R^{N_{0_{p-k}}^{(k)}} \frac{1}{(k-1)!} \right].$$

From (66),

$$c_{0_{p-1}}^{(1)} = L_{0_{p-1}}^{(1)} + P_{0_{p-1}}^{(1)} + N_{0_{p-1}}^{(1,1)} - \sum_{k=2}^p N_{0_{p-k}}^{(k,1)} \frac{1}{(k-1)!} + R^{N_{0_{p-1}}^{(1)}} + R$$

because of (69) and (70). Next,

$$c_{0_{p-2}}^{(2)} = L_{0_{p-2}}^{(2)} + P_{0_{p-2}}^{(2)} + N_{0_{p-2}}^{(2,1)} - \sum_{k=3}^p N_{0_{p-k}}^{(k,2)} \frac{1}{(k-1)!} + R^{N_{0_{p-2}}^{(2)}}$$

for the fact (69) and (71).

Because, for $k \geq 2$,

$$\begin{aligned} c_{0_{p-k}}^{(k)} &= -(p-k+1) c_{0_{p-k+1}}^{(k-1)} + L_{0_{p-k}}^{(k)} + P_{0_{p-k}}^{(k)} + N_{0_{p-k}}^{(k)} + R \\ &= (-1)^{k-1} \frac{(p-1)!}{(p-k)!} \left[L_{0_{p-1}}^{(1)} + P_{0_{p-1}}^{(1)} + N_{0_{p-1}}^{(1)} \right] + (-1)^{k-2} \frac{(p-2)!}{(p-k)!} \left[L_{0_{p-2}}^{(2)} + P_{0_{p-2}}^{(2)} + N_{0_{p-2}}^{(2)} \right] \\ &\quad + (-1)^{k-3} \frac{(p-3)!}{(p-k)!} \left[L_{0_{p-3}}^{(3)} + P_{0_{p-3}}^{(3)} + N_{0_{p-3}}^{(3)} \right] + \cdots + (-1)^{\frac{(p-k+1)!}{(p-k)!}} \left[L_{0_{p-k+1}}^{(k-1)} + P_{0_{p-k+1}}^{(k-1)} + N_{0_{p-k+1}}^{(k-1)} \right] \\ &\quad + L_{0_{p-k}}^{(k)} + P_{0_{p-k}}^{(k)} + N_{0_{p-k}}^{(k)} + R, \\ c_{0_{p+1}}^{(-1)} + \sum_{k=1}^p c_{0_{p-k}}^{(k)} \frac{1}{k!} &= \frac{1}{p} \left[- \sum_{h=1}^p P_{0_{p-h}}^{(h)} \frac{1}{(h-1)!} - N_{0_{p-1}}^{(1,1)} - \sum_{h=1}^{p-1} L_{0_{p-h}}^{(h)} \frac{1}{(h-1)!} - \sum_{h=1}^p R^{N_{0_{p-h}}^{(h)}} \frac{1}{(h-1)!} \right] \\ &\quad + \sum_{k=1}^p (-1)^{k-1} \frac{(p-1)!}{(p-k)!} \left[L_{0_{p-1}}^{(1)} + P_{0_{p-1}}^{(1)} + N_{0_{p-1}}^{(1,1)} - \sum_{h=2}^p N_{0_{p-h}}^{(h,1)} \frac{1}{(h-1)!} + R^{N_{0_{p-1}}^{(1)}} \right] \frac{1}{k!} \\ &\quad + \sum_{k=2}^p (-1)^{k-2} \frac{(p-2)!}{(p-k)!} \left[L_{0_{p-2}}^{(2)} + P_{0_{p-2}}^{(2)} + N_{0_{p-2}}^{(2,1)} - \sum_{h=3}^p N_{0_{p-h}}^{(h,2)} \frac{1}{(h-1)!} + R^{N_{0_{p-2}}^{(2)}} \right] \frac{1}{k!} \\ &\quad + \sum_{k=3}^p (-1)^{k-3} \frac{(p-3)!}{(p-k)!} \left[L_{0_{p-3}}^{(3)} + P_{0_{p-3}}^{(3)} + N_{0_{p-3}}^{(3,1)} + N_{0_{p-3}}^{(3,2)} + R^{N_{0_{p-3}}^{(3)}} \right] \frac{1}{k!} + \cdots + \\ &\quad + \sum_{k=p-1}^p (-1)^{k-p+1} \frac{(p-p+1)!}{(p-k)!} \left[L_0^{(p-1)} + P_0^{(p-1)} + N_0^{(p-1,1)} + N_0^{(p-1,2)} + R^{N_0^{(p-1)}} \right] \frac{1}{k!} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=p}^p (-1)^{k-p} \frac{(p-p)!}{(p-k)!} \left[L^{(p)} + P^{(p)} + N^{(p,1)} + N^{(p,2)} + R^{N^{(p)}} \right] \frac{1}{k!} \\
& = \left[L_{0_{p-1}}^{(1)} + P_{0_{p-1}}^{(1)} + N_{0_{p-1}}^{(1,1)} + R_{0_{p-1}}^{N^{(1)}} \right] \left(-\frac{1}{p} + \sum_{k=1}^p (-1)^{k-1} \frac{(p-1)!}{(p-k)!} \frac{1}{k!} \right) \\
& + \left[L_{0_{p-2}}^{(2)} + P_{0_{p-2}}^{(2)} + R_{0_{p-2}}^{N^{(2)}} \right] \left(-\frac{1}{p} + \sum_{k=2}^p (-1)^{k-2} \frac{(p-2)!}{(p-k)!} \frac{1}{k!} \right) \\
& + \sum_{i=3}^{p-1} \left(L_{0_{p-i}}^{(i)} + P_{0_{p-i}}^{(i)} + R_{0_{p-i}}^{N^{(i)}} \right) \left(-\frac{1}{p} \frac{1}{(i-1)!} + \sum_{k=i}^p (-1)^{k-i} \frac{(p-i)!}{(p-k)!} \frac{1}{k!} \right) \\
& + \sum_{i=2}^p N_{0_{p-i}}^{(i,1)} \left(-\sum_{k=1}^p (-1)^{k-1} \frac{(p-1)!}{(p-k)!} \frac{1}{k!} \frac{1}{(i-1)!} + \sum_{k=i}^p (-1)^{k-i} \frac{(p-i)!}{(p-k)!} \frac{1}{k!} \right) \\
& + \sum_{i=3}^p N_{0_{p-i}}^{(i,2)} \left(-\sum_{k=2}^p (-1)^{k-2} \frac{(p-2)!}{(p-k)!} \frac{1}{k!} \frac{1}{(i-1)!} + \sum_{k=i}^p (-1)^{k-i} \frac{(p-i)!}{(p-k)!} \frac{1}{k!} \right) \\
& + \left[P^{(p)} + R^{N^{(p)}} \right] \left(-\frac{1}{p} \frac{1}{(p-1)!} + \frac{1}{p!} \right) + R = R_{0_{p+1}} \text{ due to Lemma A4 where } m = 0. \blacksquare
\end{aligned}$$

Proof of Theorem 3

Proof. The constant term of $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ is zero because the constant and first order terms of $C_X^{(j-1, -1)}$ and the constant term of $C_X^{(j_0, 0)}$ are zero. Combining Lemma A1, Lemma A2, Lemma A3, Lemma A5, and Lemma A6, it is clear that all terms in $\tilde{l}_X^{(\infty)}(t, x | t_0, x_0)$ are indeterminate terms free. \blacksquare

Proof of Theorem 4

Proof. We will prove the irreducible case. Removing additional notations for Taylor expansion in x from this proof gives the proof of the reducible case. Taylor-expanding $\exp(\tilde{l}_X^{(K)})$ in (t, x) about (t_0, x_0) yields $\tilde{p}_X^{(K)}$, which satisfies the Kolmogorov backward equation

$$-\frac{\partial p_X(t, x | t_0, x_0)}{\partial t_0} = \sum_{i=1}^m \mu_i(t_0, x_0) \frac{\partial p_X(t, x | t_0, x_0)}{\partial x_{0i}} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m v_{ij}(t_0, x_0) \frac{\partial^2 p_X(t, x | t_0, x_0)}{\partial x_{0i} \partial x_{0j}}. \quad (74)$$

Because the coefficients $C_X^{(j_k, k)}$, $k = -1, 0, \dots, K$ are found after matching the same order terms of left- and right-hand sides of (74) so that its approximation error can be $O(\Delta^{K+1})$,

$$\begin{aligned}
\tilde{B}_X^{(K)}(t, x | t_0, x_0) &= -\frac{\partial \tilde{p}_X^{(K)}(t, x | t_0, x_0)}{\partial t_0} - \sum_{i=1}^m \tilde{\mu}_i(t_0, x_0) \frac{\partial \tilde{p}_X^{(K)}(t, x | t_0, x_0)}{\partial x_{0i}} \\
&\quad - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \tilde{v}_{ij}(t_0, x_0) \frac{\partial^2 \tilde{p}_X^{(K)}(t, x | t_0, x_0)}{\partial x_{0i} \partial x_{0j}}
\end{aligned} \quad (75)$$

is the remainder term. In fact $\tilde{B}_X^{(K)}(t, x | t_0, x_0) = \Delta^K \tilde{p}_X^{(K)}(t, x | t_0, x_0) \tilde{\psi}_X^{(K)}(t, x | t_0, x_0)$ where $\tilde{\psi}_X^{(K)}(t, x | t_0, x_0)$ is a sum of products of $\tilde{\mu}_i$, \tilde{v}_{ij} , $C_X^{(j_k, k)}$, $k \geq -1$ and their derivatives. $\tilde{\psi}_X^{(K)}(t, x | t_0, x_0)$ has at most polynomial growth because $C_X^{(j_k, k)}$ is determined by $\tilde{\mu}_i$ and $\tilde{\sigma}_{ij}$ that exhibit polynomial growth due to Assumption 4. $\tilde{\psi}_X^{(K)}(t, x | t_0, x_0) = O(1)$ uniformly for all (t_0, x_0) and (t, x) in a compact subset of the interior of $[0, \infty) \times S_X$ and for all θ in Θ for the reason that μ_i , σ_{ij} , and their derivatives are continuous.

Define $\tilde{r}_X^{(K)}(t, x | t_0, x_0) = \tilde{p}_X^{(K)}(t, x | t_0, x_0) - p_X(t, x | t_0, x_0)$ then $\tilde{r}_X^{(K)}(t, x | t_0, x_0)$ also satisfies (74) with the same remainder $\tilde{B}_X^{(K)}(t, x | t_0, x_0)$ since (74) holds true for $p_X(t, x | t_0, x_0)$. In addition, $\tilde{r}_X^{(K)}(t, x | t_0, x_0) \rightarrow 0$

as $\Delta \rightarrow 0$ on account of the fact that both $\tilde{p}_X^{(K)}(t, x|t_0, x_0)$ and $p_X(t, x|t_0, x_0)$ converge to a Dirac mass at (t_0, x_0) as $\Delta \rightarrow 0$. As a matter of fact

$$\tilde{r}_X^{(K)}(t, x|t_0, x_0) = \int_{S_X} \int_t^{t_0} \tilde{B}_X^{(K)}(t, x | \tau, z) p_X(0, z|\tau - t_0, x_0) d\tau dz \quad (76)$$

is the solution because (76) satisfies the initial boundary condition and (75). As can be seen below, the tails of p_X decrease at an exponential rate in a neighborhood of $\Delta = 0$ while $\tilde{B}_X^{(K)}$ has polynomial growth. Hence $\tilde{B}_X^{(K)} p_X$ is integrable and $\tilde{r}_X^{(K)}(t, x|t_0, x_0) = O(\Delta^K)$ uniformly for all (t_0, x_0) and (t, x) in a compact subset of the interior of $[0, \infty) \times S_X$ and for all θ in Θ . Let

$$\tilde{R}_X^{(K)}(t, x|t_0, x_0) \equiv \sup_{\theta \in \Theta} \left| \tilde{r}_X^{(K)}(t, x|t_0, x_0) \right|.$$

Then

$$\begin{aligned} E \left[\tilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \middle| X_0 = x_0 \right] &= \int_{S_X} \tilde{R}_X^{(K)}(t, x|t_0, x_0) p_X(t, x|t_0, x_0) dx \\ &= \int_N \tilde{R}_X^{(K)}(t, x|t_0, x_0) p_X(t, x|t_0, x_0) dx \\ &\quad + \int_{N \setminus S_X} \tilde{R}_X^{(K)}(t, x|t_0, x_0) p_X(t, x|t_0, x_0) dx, \end{aligned}$$

where $N = \prod_{i=1}^m [x_{0i} - \sqrt{\Delta} c_{\Delta}, x_{0i} + \sqrt{\Delta} c_{\Delta}]$ is a neighborhood of x_0 , $N \setminus S_X$ is its complement and c_{Δ} is a sequence of positive numbers such that $c_{\Delta} \rightarrow \infty$ and $\sqrt{\Delta} c_{\Delta} \rightarrow 0$.

Because $\tilde{r}_X^{(K)}(t, x|t_0, x_0) = O(\Delta^K)$, $\tilde{R}_X^{(K)}(t, x|t_0, x_0) \leq M \Delta^K$ for some constant M and for all $x \in N$.

Thus

$$\int_N \tilde{R}_X^{(K)}(t, x|t_0, x_0) p_X(t, x|t_0, x_0) dx = O(\Delta^K).$$

In a neighborhood of $\Delta = 0$, the tail behavior of p_X is driven by the term $\exp \left[-\frac{m}{2} \ln(\Delta) + C_X^{(j-1, -1)}(t, x|t_0, x_0) \Delta^{-1} \right]$, with $C_X^{(j-1, -1)}(t, x|t_0, x_0) = -(1/2)(x - x_0)^T v^{-1}(t_0, x_0)(x - x_0) + o(\Delta)$. Because $\tilde{R}_X^{(K)}$ grows at a polynomial rate, the expected value of $\tilde{R}_X^{(K)}$ outside of N involves integrating $\|x - x_0\|^b$, $b \geq 0$ against the exponential tails of p_X . Considering the $m = 1$ case for simplicity, on the interval $(\sqrt{\Delta} c_{\Delta}, \infty)$, it is of the form

$$\sqrt{\Delta} \int_{\sqrt{\Delta} c_{\Delta}}^{\infty} |x - x_0|^b \exp \left[-\frac{(x - x_0)^2}{2\Delta v(t_0, x_0)} \right] dx = \Delta^{b/2} \int_{c_{\Delta}}^{\infty} |z - z_0|^b \exp \left[-\frac{(z - z_0)^2}{2v(t_0, x_0)} \right] dz$$

by the change of variable $z - z_0 = (x - x_0)/\sqrt{\Delta}$ and similarly on the interval $(-\infty, -\sqrt{\Delta} c_{\Delta})$. These integrals converges to zero since $c_{\Delta} \rightarrow \infty$. Hence $E \left[\tilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \middle| X_{t_0} = x_0 \right] \rightarrow 0$ as $\Delta \rightarrow 0$.

Following the similar procedure we can prove that $\text{Var} \left[\tilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \middle| X_{t_0} = x_0 \right] \rightarrow 0$ as $\Delta \rightarrow 0$. Combining these two results, given X_{t_0} , $\tilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \rightarrow 0$ in P_{θ_0} -probability due to the Chebyshev's inequality. For a positive number ε ,

$$P \left(\left| \tilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \right| > \varepsilon \right) = \int_{S_X} P \left(\left| \tilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \right| > \varepsilon \middle| X_{t_0} = x_0 \right) \pi_{t_0}(x_0) dx_0,$$

where $\pi_{t_0}(x_0)$ is the marginal density of X_t at time t_0 . Notice that $0 \leq P(\cdot) \leq 1$ and the density function $\pi_{t_0}(x_0)$ integrates to one. So, applying Lebesgue's dominated convergence theorem $\tilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \rightarrow 0$ in P_{θ_0} -probability without conditioning on X_{t_0} .

Because of the convergence of $\tilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0})$ to 0 in P_{θ_0} -probability and the fact that the logarithm is continuous, $\tilde{l}_X^{(K)}(\theta) \rightarrow l_X(\theta)$ in P_{θ_0} -probability and so does $\tilde{l}_n^{(K)}(\theta)$ to $l_n(\theta)$ for fixed n . As a result, the maximizer of $\tilde{l}_n^{(K)}(\theta)$, $\hat{\theta}_{n,\Delta}^{(K)} \in \Theta$, exists almost surely since the maximizer $\hat{\theta}_{n,\Delta}$ of $l_n(\theta)$ is assumed to exist. And they are close to each other as $\Delta \rightarrow 0$ in the sense that $\hat{\theta}_{n,\Delta}^{(K)} \rightarrow \hat{\theta}_{n,\Delta}$ in P_{θ_0} -probability. Taking $\Delta \rightarrow 0$ sufficiently fast, the speed at which $\hat{\theta}_{n,\Delta}^{(K)} \rightarrow \hat{\theta}_{n,\Delta}$ can be made arbitrarily high for any n . Hence a sequence $\Delta_n \rightarrow 0$ can be taken so that (41) is satisfied. ■